Aggregate growth models attempt to highlight particular elements of a very complex growth process, always by making some drastic simplifying assumptions. Each new generation of growth models attempts to fix some flaw in previous models, but often end up create new paradoxes. Solow's 1956 "Contribution to the theory of economic growth" for example criticized and "fixed" the fixed proportions assumption of the Harrod-Domar (HD) model. But in doing so he replaced the HD growth model with one of "steady-state" income levels and "exogenous" growth. Similarly, Romer (1986) and Lucas (1988) attempt to set the stage for a new generation of "endogenous" growth models by pointing out the limits of the Solow-Swan model, but tractable endogenous growth models often end up with constant returns to producible inputs (as with the HD model). The conditional convergence implied by the augmented Solow model has policy implications similar to those poverty trap models (fix a few things, then you will grow). Each break with the past reveals yields new insights, but also much common ground. Each generation of growth models complements rather than replaces the previous generation; each allows use to view the complex growth and development process from a different and generally empirically testable perspective. An example is the endogenous growth literature of the 1990s, which offered a radical break with the past. But a series of "hybrid" models show these two perspectives are not as different as once assumed, hence we begin with three growth models, rather than one.

Generation 1: Harrod Domar (HD) growth model, γ = sA where s is the savings rate and the fixed A is average and marginal product of capital: The HD model starts with a few national accounting identities and some strong assumptions, but still ends up with some interesting policy implications. Starting with the national accounts identity Y = C + I (ignoring government and trade) Y, I and C are determined as in Hick's ISLM setup assuming a constant marginal propensity to consume or equivalently a fixed savings rate such that I = sY and C = (1-s)Y. Investment or I is determined by exogenous "animal spirits" or by using a simple accelerator model (increasing sales increases investment). Either way, once we know I (or I and G) we know C and Y: $Y^d = (1/s)I$ where 1/s is the familiar "multiplier." Supply or Y^s is determined by the simplest possible production function Y = AK where A is the fixed output to capital ratio, as clearly A=Y/K. Or $Y^s = A(K+I)$ if there is no "time to build." Note that A is the average and the marginal product of capital (MPK). The famous incremental capital output ratio or ICOR is 1/A. To complete the HD growth model we just need a link between investment I and K the total capital stock. Assuming no depreciation of capital ΔK = I = sY = sAK. Noting that ΔY = A*ΔK and using the fact that ΔK = I = sY yields ΔY = AsY yielding,

$$\frac{\Delta Y}{Y} = sA \ or \ \ \gamma = sA.$$

Demand: (1) $Y^d = C + I$ where I is determined by an investment function and C = (1-s)Y so that $Y^d = (1/s)I = (1/s)\Delta K$ where we assume the depreciation of capital is zero. **Supply:** (2) $Y^s = AK$. Perhaps the most famous and surprising implication of the HD model is that the economy is dynamically unstable. A small in creare in investment from its "warranted rate" for example creates a permanent excess of demand over supply, so the economy enters an inflationary spiral. A small decrease in demand leads to a deep recession with high unemployment (though the labor market is not modeled explicitly).

The HD model ignores labor as input and use a very simple fixed coefficients technology. In the mid 1950s Robert Solow looked outside and saw a fairly stable growing capitalist economy with a tightening labor market and then wrote down the model that won him a noble prize (though it now known as the Solow/Swan model). The modifications Solow made to the HD model seem obvious in retrospect, the implications of assuming diminishing returns of capital to some exogenously determined input (scarce labor) was not obvious. In fact growth model is in something of a misnomer: the Solow-Swan framework really gave us a model of income levels, or steady states. The part of growth that is endogenous to the Solow model is transitional: unless one assumes exogenous technical change or productivity, growth ends once the steady state is reached.

Generation 2: The Solow or Neoclassical model with a fixed savings rate implies a long run growth rate of $\gamma = n + \lambda$ where n and λ are the exogenous rates of population growth and technical change.

The Solow-Swan model adds a labor input that can be substituted for capital. With the a Cobb-Douglas production function for example, $Y = A(t)K^{\alpha}L^{\beta}$ where A(t) represents the exogenous rate of technological change (growing at λ percent per year) and α is the share of capital in national income. The convention is to rewrite the model in labor units so that in the constant returns or Cobb-Douglas case α + ß = 1 so that $y = A(t)k^{\alpha}$ where k / K/L and y / Y/L. Solow assumes a fixed savings share s, as in the HD model implying savings (investment) per worker is $sy = sAk^{\alpha}$. Since demand for capital grows at nk and the supply of savings (investment) per worker is sy, the critical equation for the change in capital k and output per worker y is,

$$\dot{k} = sy - nk \quad or \quad \dot{k} = sk^{\alpha} - nk \quad which\ implies\ \frac{\dot{k}}{k} = sk^{\alpha-1} - n \quad . \qquad \textbf{(3)}$$

The Solow model avoids the HD dynamic instability problem. To see this note that when sy > nk there is an excess supply of capital the price of K falls and it is substituted for labor raising k, and vice versa. When sy = nk and A(t) is fixed at 1 (unless it changes exogenously at rate λ). This implies the steady state $k^* = (s/n)^{\mu}$ where $\mu = 1/(1-\alpha)$. At this point per capita income changes only if there is exogenous technical change that is, if λ > 0. Recall that $y = k^{\alpha}$ and set λ = 0. Sometimes (some authors, including Sachs et al., 2004 include depreciation, d, and/or a positive exogenous rate of productivity growth or technical change in A(t), λ > 0. This implies equation (3) is,

$$\dot{k} = sy - (n + d + \lambda)k;$$

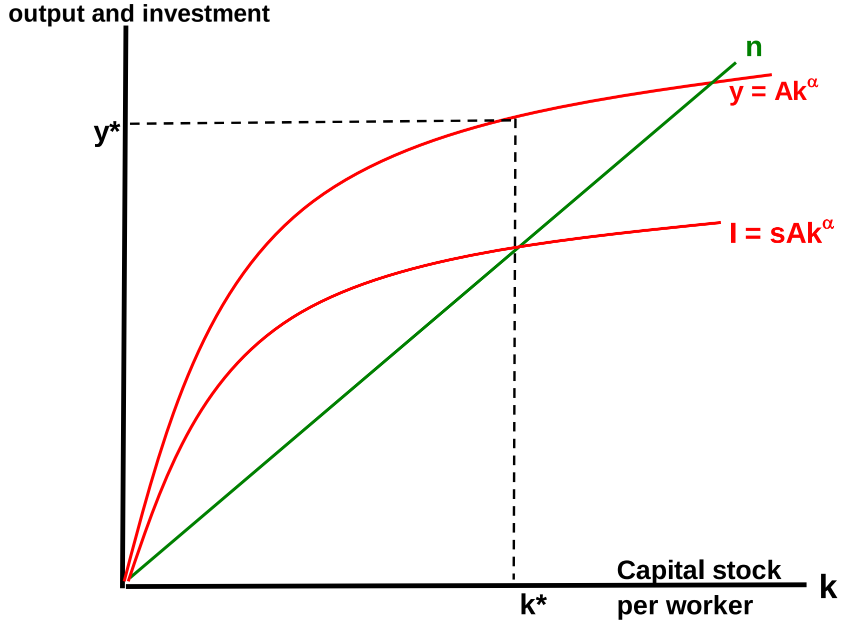
$$\dot{k} = sAk^{\alpha} - (n + d + \lambda)k \quad so\,that\ \frac{\dot{k}}{k} = sAk^{\alpha-1} - (n + d + \lambda)n$$

$$yielding\ a\ steady\ state\ k^* = \left(\frac{sA}{(n + d + \lambda)}\right)^{1/(1-\alpha)}$$

Clearly the capital per worker ratio k* increases with the savings rate and productivity level A, leading to a burst of growth until the higher steady state is reached. Similarly, an increase in n or d and even λ reduces the steady-state k*. The last result is a bit counter-intuitive until you realize that A represents x efficiency, something for nothing, so if it becomes easier to raise the productivity of labor via technical change, why do it the hard way, by saving more and raising capital per worker?

The standard Solow model diagram is for income levels (we switch to a growth rate diagram below) as show in Figure 1. There is only one "steady state" level of income generated by k*, that is y* = $A(k^*)^\alpha$. The single steady state however just reflects our assumption that neither n, s or A depends on the level of income y. Mathematically we get this nice shape when we both employ a Cobb-Douglas production function and what are called the Inada conditions (defined below). If we relax these assumptions and more importantly let savings or productivity A or population growth depend on income (which is a plausible) then we may have poverty traps and multiple equilibria, as in Sachs et al. 2004 and discussed further below.

Figure 1 Solow Model Income Levels Diagram



Generation 3: Endogenous growth or "AK" model with steady state growth rate: γ = (1/θ)(A - ρ) Starting again with Y = AK and Y = C + I so that per capita investment is I = Ak - c we assume K includes both physical and human capital. The key assumption of endogenous growth models is that even if labor and natural resources are in limited supply, growth can proceed without them using only produced inputs: there are constant returns to produced inputs. An upgrade from the HD model is that endogenous growth models solve for the Ramsey optimal savings rate assuming maximize a constant relative risk aversion utility function along the lines of $U(C) = \frac{(C^{(1-\theta)} - 1)}{1-\theta}$

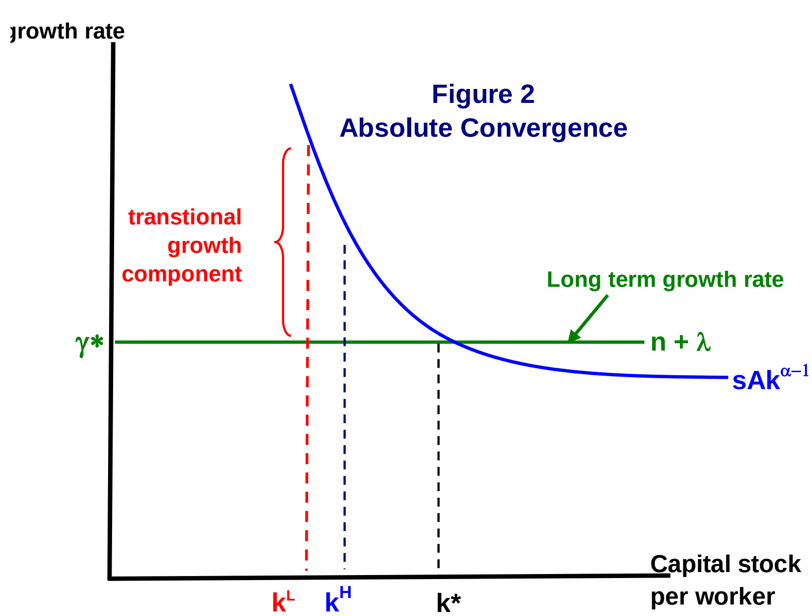
where 1/θ is the constant intertemporal elasticity of substitution (IES) and ρ is the discount rate (see the full Barro Government and Growth handout for a formal derivation of this model-- since the MPK is constant at A, it is important that IES be constant as well). In this new setup the savings rate depends on the parameters σ and ρ which in turn depend on household preferences and under some interpretations, population growth (since how one values the consumption in the future depends in part on how one values the consumption of children). As 1/σ falls or ρ rises people to prefer consumption today over consumption tomorrow implying they save less and growth falls. An important empirical implication of many (but not all) endogenous growth models is that growth rates need show no convergence across countries and that the endogenous savings rate (or the parameter which affect time preference) is an important determinant of long term growth rate. Note also that this model has a steady-state growth rate, but no steady income level (income grows forever). This class of models is consistent with the roughly constant rate of per capita growth observed over long periods in countries like the U.S., but this result relies on strong and key assumption: constant returns to produced factors. Switching to Stone-Geary preferences: (see page 27 of Robelo, 1992)

$U(C)=\frac{(C-\bar{C})^{(1-\theta)}-1}{1-\theta}$ yields a low income poverty trap, where $\bar{C}$ is subsistence consumption: as C falls toward subsistence $\bar{C}$ the IES or 1/θ approaches zero and savings goes to zero creating a low savings poverty trap.

Growth Models with and without Poverty traps:

Perhaps the best way to gain some intuition about these models is by putting them into diagrams. The most famous diagram is that of Solow (1956) it plots investment and the level of output y against the capital stock per worker, as shown in Figure 1 below. Two solve this model (or any growth model) we are generally concerned about the existence of a unique solution (steady state in this case) and the stability of that equilibrium or steady state. Recall that in the HD model we get a unique solution, but it is not stable. As it turns out, poverty trap models generally do not have a unique solution, rather there are two possible outcomes: the low level equilibrium trap and the traditional higher income steady state. One way to assure a single unique solution (and rule out many poverty traps) is to invoke the so called Inada conditions on what happens to the MPK when capital per worker gets very small or very large. As the capital stock approaches zero, the marginal product of capital (MPK) approaches infinity, $\lim_{k\to 0} f'(k)\to\infty$ and as k becomes very large, the MPK approaches zero, that is $\lim_{k\to\infty} f'(k)\to 0$. Taken together the Inada conditions give the investment curve in the Solow model diagram the shape shown below in Figure 1. The slope of the red lines reflects the MPK, the fact that it is very steep means these curves are very steep near the origin, always above the green population capital demand curve with slope n (the rate of population growth). However as k increases the MPK starts to fall towards zero, meaning at some point it must cross the green population line (slope n) from above, assuring us the single steady state k*, y* we seek, as shown in Figure 1.

Absolute Convergence: Return to (3) above we can rewrite the key dynamic equation as a growth rate by dividing through by k, $\frac{\dot{k}}{k}=sAk^{\alpha-1}-n$ where if $\frac{\dot{k}}{k}=0$, then $\frac{\dot{y}}{y}=0$ or at least the increment to y caused by an increase in k goes to zero. Growth may continue due exogenous technical change, where $\frac{\dot{A}}{A}=\lambda$ and assuming capital does not depreciate at rate d or δ we obtain an exogenous growth rate of per capita y of γ determined by n and λ both of which are determined outside the model for now (hence the term exogenous growth model). The standard Solow model has one important testable implication: if two economies are heading toward the same steady state, for example k* in Figure 2, then

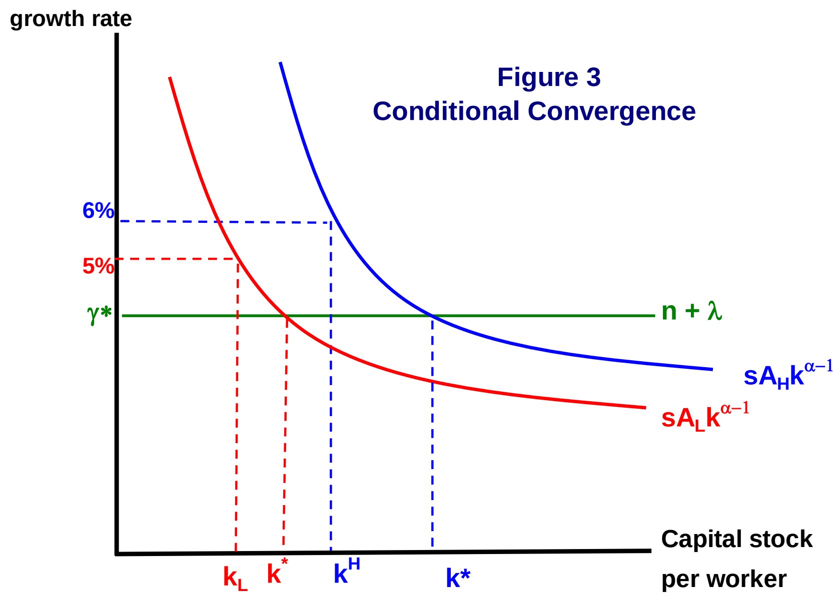


the low income economy will tend to grow faster than the high income economy until both countries reach their long run capital stock per worker k* and settle into the same long run growth rate per person λ (the exogenous rate of technical change). Note that the HD model shows no such tendency, the economy with the highest productivity savings rate combination sA will grow faster forever, there is not convergence. As it happens when array data on growth rates against initial per capital income (say 1960 or 1970) we see no such pattern: poor economies on the whole have not grown faster than richer countries.

Conditional convergence: As Lucas (1988) points out, the original Solow model cannot explain the variation in income across countries (there is not enough variation in capital per worker k) nor is it consistent with the lack of absolute convergence discussed above. The resurrection of the neoclassical growth model rests on three modifications: adding human capital to obtain the "augmented Solow" model explored by Mankiw, Romer and Weil (1992). Once human capital (typically years of education per adult) is added to a growth regression, one obtains strong evidence for conditional convergence, and a large proportion of variance in income levels across countries can be explained by human plus physical capital.[1]

Suppose k^H in Figure 3 is the steady state associated with a high level of education, whereas k_L reflects a low level of human capital (using the approach of Jones (2003) Chapter 3 page 55) in this case

$$k^H = \tilde{k}^H = \left(\frac{k}{Ah}\right) \; where \; h = e^{\psi\mu}$$, μ is average years of education and ψ reflects of the productivity of human capital. In this case if we look at the data without controlling for human capital per worker (differences in h) it will appear the rich country is growing faster than the poor country. Yet this result is still consistent with conditional convergence, we just need control for the fact that these two countries are heading for very different steady states (the red vs. the blue k*). Conditional on education (or savings rates for example) the poorer country will grow faster. Hence conditional convergence shifts the focus to determining why a given country has a higher steady state than others (e.g., higher savings rates, human capital investment, productivity, etc.).



Some growth models generate robust empirical results without a great deal of policy relevance. Conditional convergence is a good example. Income levels converge to some extent, but to different steady states. If we define

1 Sala-i-Martin (2004) for example argues evidence supporting conditional convergence demonstrates the validity of the augmented Solow model. But some hybrid models also show weak convergence, though growth in endogenous (as with the Sobelo model discussed below). Some empirical regularities are robust empirically (conditional convergence for example) but don't have great policy relevance. Other models with great policy relevance, poverty traps for example, are hard to find in the empirical data.

steady states ex-post, in advertently, or pick a correlate as opposed to a cause of high steady state income, we may not learn much of policy relevant. Strong absolute convergence has strong policy implications for government: just get out of the way and let diminishing returns to labor work its magic. Relatively weak though robust conditional convergence does not have strong result for policy, especially when there is some evidence higher incomes cause education, the reverse of the MRW result. It is 2004 survey Sala-i-Martin argues "that the conditional convergence hypothesis is one of the strongest and most robust empirical regularities found in the data. Hence, by taking the theory seriously, researchers arrived at the exact opposite empirical conclusion: the neoclassical model is not rejected by the data, whereas the AK model is." (Sala-i-Martin, 2004 p. 44). However on the next page he grants that many endogenous growth models imply conditional convergence as well (including some hybrid models discussed in Barro and Sala-i-Martin, 2004). Pritchett (2005) for example argues the return on education investment has been disappointing; one interpretation of this is that causality may run from income to education. Similar evidence has been accumulated pointing to the dependence of savings rates on income and growth, as opposed to the reverse argument. Other models with great policy relevance, poverty traps for example, are hard to find empirical evidence for. Other models with strong policy implications, poverty traps for example, are hard to find support for empirically. To the extent that development involves more than growth, and countries seem to get stuck at the low levels of development, the concept of multiple equilibria has strong appeal.

Poverty Traps

The Inada conditions are plausible and more or less guarantee one equilibrium steady state income level, but they need not always hold. In fact when growth stagnated or even reversed in Africa during the 1980s and 1990s, Sachs et al. (2004) argue governance indicators alone cannot explain Africa's underperformance. They use a Solow model diagram very similar to those discussed above to illustrate three possible poverty traps that lead to multiple equilibria. Their baseline Figure 1 is similar to our Figure 1 above, except they allow for depreciation of capital plus population to drive the demand for capital d+n. And they use y=f(k) instead of $y = Ak^{\alpha}$ (the Cobb-Douglas case discussed above). The three poverty traps make on of the p+arameters of the Solow model endogenous:

1. The minimum capital stock makes productivity depend on the level or the capital stock, that is A(k) instead of just A(t) as in the standard model (recall that A(t) grows at exogenous rate λ).

2. In a savings trap, the savings rate s(y) rises with income but s goes to zero as consumption reaches some minimum subsistence lever (as in the Stone-Geary utility function discussed above see their Figure 3)

3. With demography trap, population growth is a function of income per worker, n(y), when income is too low population growth is very rapid, as income rises, n falls.

In all three cases a new level in addition to the stable steady state K_E (our k*) discussed above. Sachs et al. (2004) label this threshold level of capital stock K_T. Once beyond this threshold, the economy converges automatically to the higher (world?) steady state. Below this threshold, the capital stock per worker falls steadily plunging the economy into poverty, hence the term “poverty trap” if you cannot reach the threshold level of capital per worker (income) the economy remains stuck in poverty.

Problems and Examples:

1. Dynamic instability: The HD model is famously dynamically unstable in that a small deviation from its “warranted” growth path leads to major booms and busts. To illustrate the “razor’s-edge” problem set $s = .2$, $A = .333$ and $K = 120$ then compute $Y^d = (1/s)*I$ and $Y^s = AK$ for $I = 8$. What is the growth rate $g = \Delta K/K = \Delta Y/Y$ where $I = \Delta K$? This is the so-called warranted growth rate that makes $Y^d = Y^s$. Suppose investors become optimistic about the future and choose $I = 10$. What is the new Y^d compared to $Y^s = AK$ or $Y^s = A(K+I)$? What message does the resulting supply and demand imbalance send to firms? How would they respond? Try the same exercise with $I = 5$. How do these examples illustrate the “razor’s edge” problem? (see also Chaing p. 468).

2. **The speed of convergence:** When k is some distance away from its steady state value catch up growth can be quite dramatic. With $s = .12$, $n = .03$ and $\alpha = .5$ the steady state capital stock k^* is 16. What is the rate of change in k when the country has not yet reached its steady state k^*? Now try $k = 16$, then $k = 4$, 8 or 12.

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Figure 1. Standard Neoclassical Growth Model

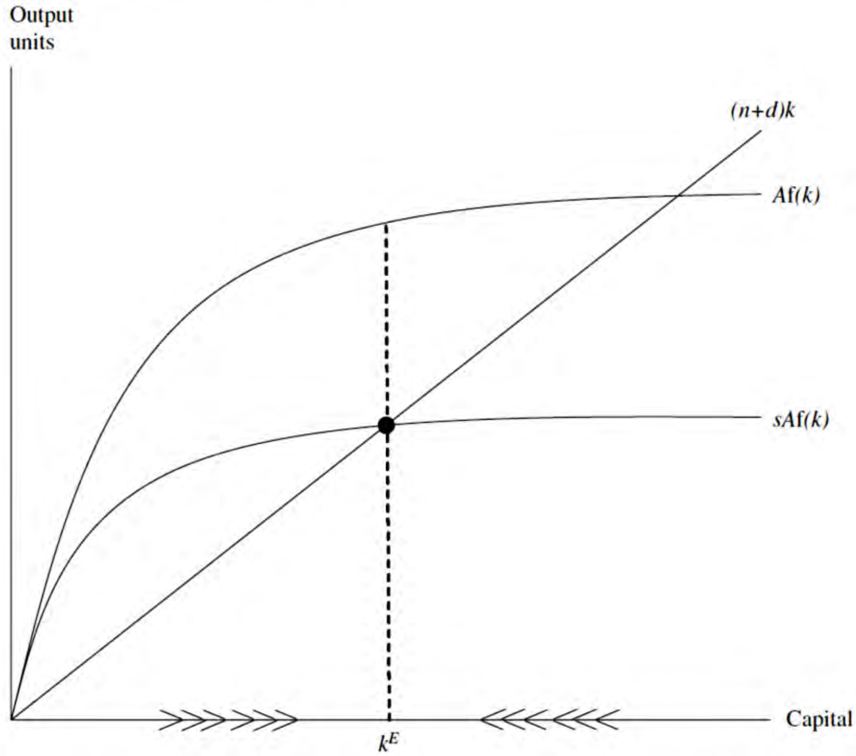


Figure 2. Growth Model with Minimum Capital Stock Threshold

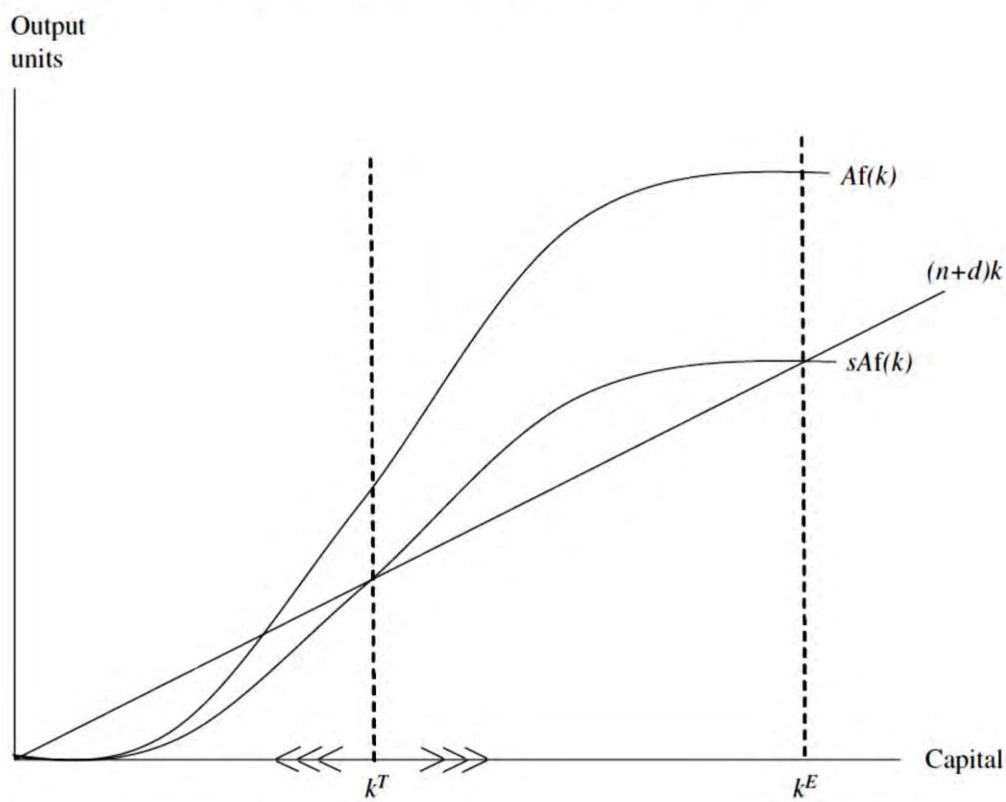


Figure 3. Saving Trap

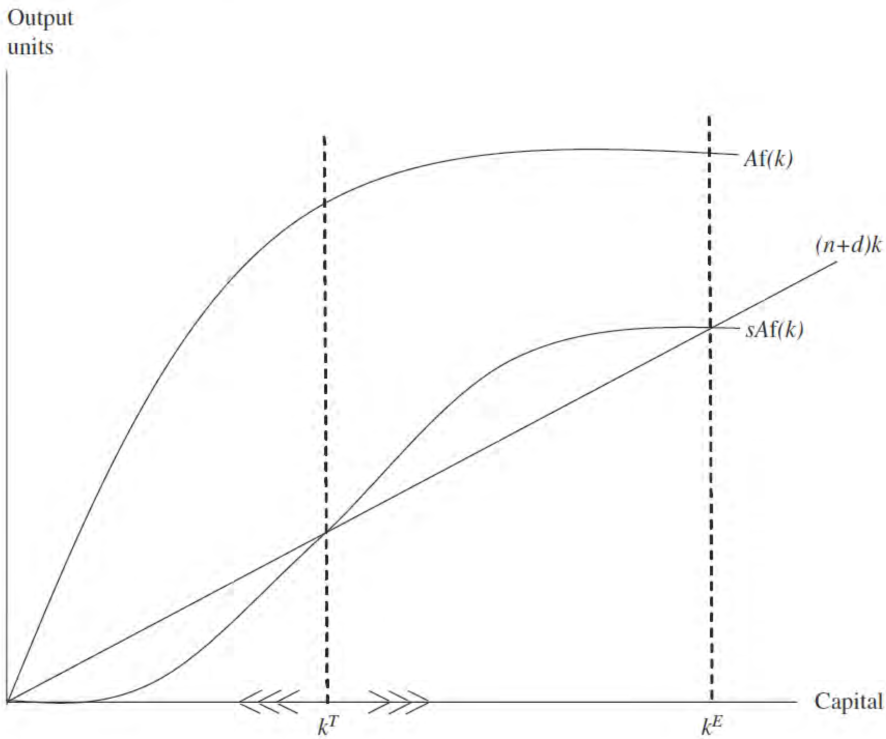
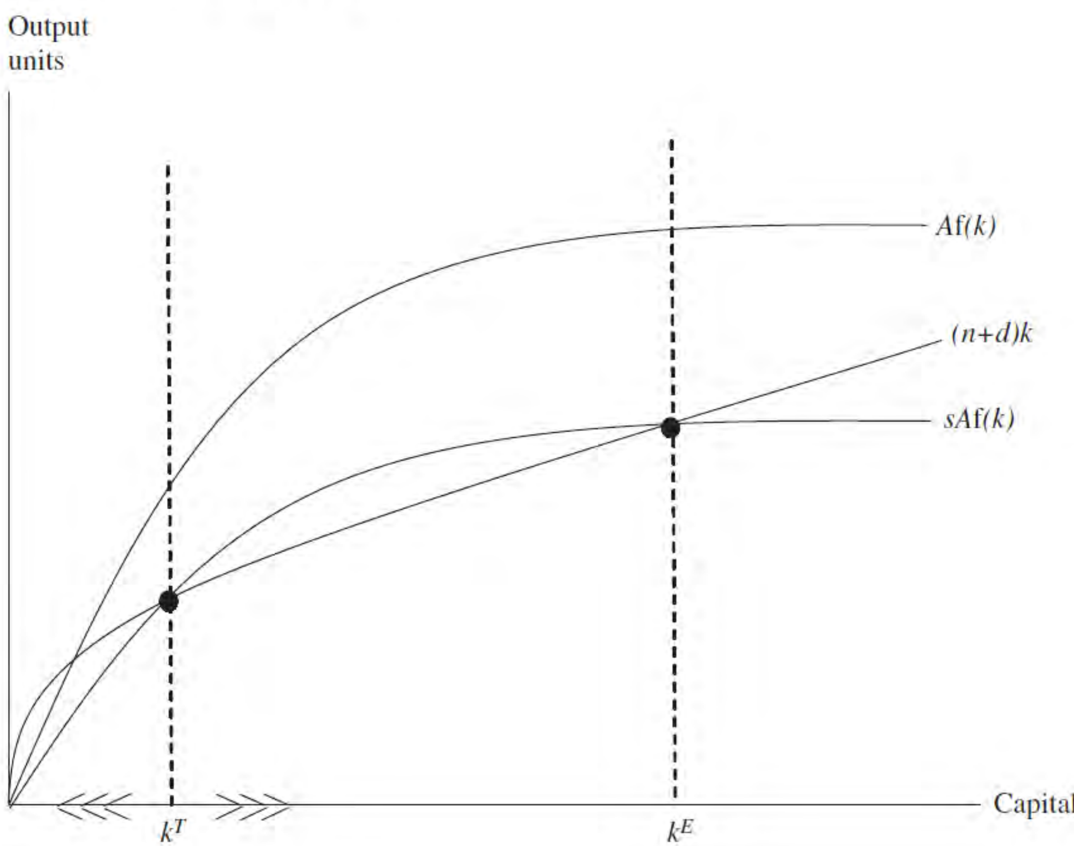
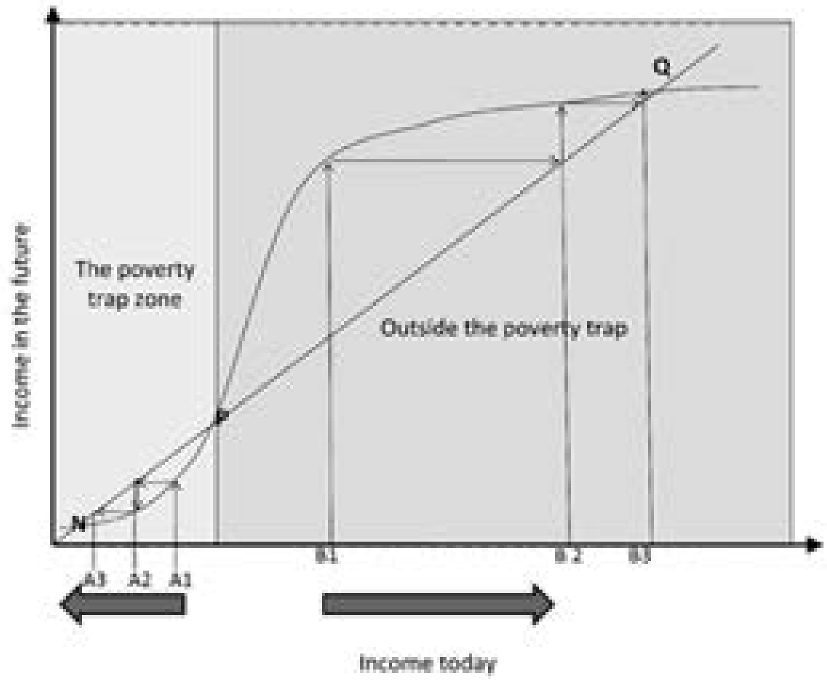


Figure 4. Demographic Trap



This poverty trap diagram is discussed in the Banerjee and Duflo, 2011 book poor economics (and now on the "Why Poverty" web page). It is more general in the sense that almost any cumulative growth process (poverty trap) is consistent with this diagram, in this sense the Solow-Swan model is a specific example of what may be more general phenomenon.



S-Shape Curve and the Poverty Trap