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Introduction To Economic Growth 2nd Edition

Chapter 2
The Solow Model

> All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive. —ROBERT SOLOW (1956), P. 65.

In 1956, Robert Solow published a seminal paper on economic growth and development titled "A Contribution to the Theory of Economic Growth." For this work and for his subsequent contributions to our understanding of economic growth, Solow was awarded the Nobel Prize in economics in 1987. In this chapter, we develop the model proposed by Solow and explore its ability to explain the stylized facts of growth and development discussed in Chapter 1. As we will see, this model provides an important cornerstone for understanding why some countries flourish while others are impoverished.

Following the advice of Solow in the quotation above, we will make several assumptions that may seem to be heroic. Nevertheless, we hope that these are simplifying assumptions in that, for the purposes at hand, they do not terribly distort the picture of the world we create. For example, the world we consider in this chapter will consist of countries that produce and consume only a single, homogeneous good (*output*). Conceptually, as well as for testing the model using empirical data, it is convenient to think of this output as units of a country's gross domestic product, or GDP. One implication of this simplifying assumption is

that there is no international trade in the model because there is only a single good: I'll give you a 1941 Joe DiMaggio autograph in exchange for... your 1941 Joe DiMaggio autograph? Another assumption of the model is that technology is *exogenous*—that is, the technology available to firms in this simple world is unaffected by the actions of the firms, including research and development (R&D). These are assumptions that we will relax later on, but for the moment, and for Solow, they serve well. Much progress in economics has been made by creating a very simple world and then seeing how it behaves and misbehaves.

Before presenting the Solow model, it is worth stepping back to consider exactly what a model is and what it is for. In modern economics, a model is a mathematical representation of some aspect of the economy. It is easiest to think of models as toy economies populated by robots. We specify exactly how the robots behave, which is typically to maximize their own utility. We also specify the constraints the robots face in seeking to maximize their utility. For example, the robots that populate our economy may want to consume as much output as possible, but they are limited in how much output they can produce by the techniques at their disposal. The best models are often very simple but convey enormous insight into how the world works. Consider the supply and demand framework in microeconomics. This basic tool is remarkably effective at predicting how the prices and quantities of goods as diverse as health care, computers, and nuclear weapons will respond to changes in the economic environment.

With this understanding of how and why economists develop models, we pause to highlight one of the important assumptions we will make until the final chapters of this book. Instead of writing down utility functions that the robots in our economy maximize, we will summarize the results of utility maximization with elementary rules that the robots obey. For example, a common problem in economics is for an individual to decide how much to consume today and how much to save for consumption in the future. Another is for individuals to decide how much time to spend going to school to accumulate skills and how much time to spend working in the labor market. Instead of writing these problems down formally, we will assume that individuals save a constant fraction of their income and spend a constant fraction of their time accumulating skills. These are extremely useful simplifications; without them, the models are difficult to solve without more advanced

mathematical techniques. For many purposes, these are fine assumptions to make in our first pass at understanding economic growth. Rest assured, however, that we will relax these assumptions in Chapter 7.

2.1 THE BASIC SOLOW MODEL

The Solow model is built around two equations, a production function and a capital accumulation equation. The production function describes how inputs such as bulldozers, semiconductors, engineers, and steelworkers combine to produce output. To simplify the model, we group these inputs into two categories, capital, K, and labor, L, and denote output as Y. The *production function* is assumed to have the Cobb-Douglas form and is given by

$$Y = F(K, L) = K^{\alpha}L^{1-\alpha}, \tag{2.1}$$

where α is some number between 0 and 1.[1] Notice that this production function exhibits *constant returns to scale:* if all of the inputs are doubled, output will exactly double.[2]

Firms in this economy pay workers a wage, w, for each unit of labor and pay r in order to rent a unit of capital for one period. We assume there are a large number of firms in the economy so that perfect competition prevails and the firms are price-takers.[3] Normalizing the price of output in our economy to unity, profit-maximizing firms solve the following problem:

$$\max_{K,L} F(K, L) - rK - wL.$$

According to the first-order conditions for this problem, firms will hire labor until the marginal product of labor is equal to the wage and will rent capital until the marginal product of capital is equal to the rental price:

$$w = \frac{\partial F}{\partial L} = (1 - \alpha)\frac{Y}{L},$$
$$r = \frac{\partial F}{\partial K} = \alpha\frac{Y}{K}.$$

Notice that $wL + rK = Y$. That is, payments to the inputs ("factor payments") completely exhaust the value of output produced so that there are no economic profits to be earned. This important result is a general property of production functions with constant returns to scale. Notice also that the share of output paid to labor is $wL/Y = 1 - \alpha$ and the share paid to capital is $rK/Y = \alpha$. These factor shares are therefore constant over time, consistent with Fact 5 from Chapter 1.

Recall from Chapter 1 that the stylized facts we are typically interested in explaining involve output per worker or per capita output. With this interest in mind, we can rewrite the production function in equation (2.1) in terms of output per worker, $y \equiv Y/L$, and capital per worker, $k \equiv K/L$:

$$y = k^{\alpha}. \tag{2.2}$$

This production function is graphed in Figure 2.1. With more capital per worker, firms produce more output per worker. However, there are diminishing returns to capital per worker: each additional unit of capital we give to a single worker increases the output of that worker by less and less.

The second key equation of the Solow model is an equation that describes how capital accumulates. The capital accumulation equation is given by

$$\dot{K} = sY - dK. \tag{2.3}$$

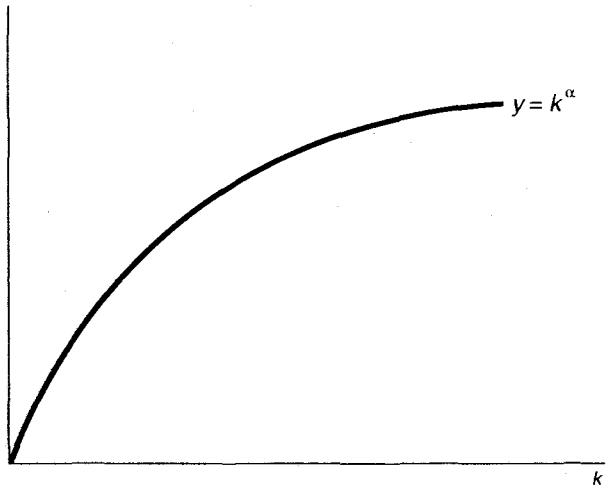
This kind of equation will be used throughout this book and is very important, so let's pause a moment to explain carefully what this equation says. According to this equation, the change in the capital stock, $\dot{K}$, is equal to the amount of gross investment, sY, less the amount of depreciation that occurs during the production process, dK. We'll now discuss these three terms in more detail.

[1] Charles Cobb and Paul Douglas (1928) proposed this functional form in their analysis of U.S. manufacturing. Interestingly, they argued that this production function, with a value for α of 1/4, fit the data very well without allowing for technological progress.

[2] Recall that if $F(aK, aL) = aY$ for any number $a > 1$, then we say that the production function exhibits constant returns to scale. If $F(aK, aL) > aY$, then the production function exhibits *increasing returns to scale*, and if the inequality is reversed the production function exhibits *decreasing returns to scale.*

[3] You may recall from microeconomics that with constant returns to scale the number of firms is indeterminate—i.e., not pinned down by the model.

FIGURE 2.1 **A COBB-DOUGLAS PRODUCTION FUNCTION**



The term on the left-hand side of equation (2.3) is the continuous time version of $K_{t+1} - K_t$, that is, the change in the capital stock per "period." We use the "dot" notation[4] to denote a derivative with respect to time:

$$\dot{K} \equiv \frac{dK}{dt}.$$

The second term of equation (2.3) represents gross investment. Following Solow, we assume that workers/consumers save a constant fraction, s, of their combined wage and rental income, $Y = wL + rK$. The economy is closed, so that saving equals investment, and the only use of investment in this economy is to accumulate capital. The consumers then rent this capital to firms for use in production, as discussed above.

The third term of equation (2.3) reflects the depreciation of the capital stock that occurs during production. The standard functional form used here implies that a constant fraction, d, of the capital stock depreciates every period (regardless of how much output is produced). For

[4]Appendix A discusses the meaning of this notation in more detail.

example, we often assume $d = .05$, so that 5 percent of the machines and factories in our model economy wear out each year.

To study the evolution of output per person in this economy, we rewrite the capital accumulation equation in terms of capital per person. Then the production function in equation (2.2) will tell us the amount of output per person produced for whatever capital stock per person is present in the economy. This rewriting is most easily accomplished by using a simple mathematical trick that is often used in the study of growth. The mathematical trick is to "take logs and then derivatives" (see Appendix A for further discussion). Two examples of this trick are given below.

Example 1:

$$k \equiv K/L \Longrightarrow \log k = \log K - \log L$$
$$\Longrightarrow \frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}.$$

Example 2:

$$y = k^{\alpha} \Longrightarrow \log y = \alpha \log k$$
$$\Longrightarrow \frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k}.$$

Applying Example 1 to equation (2.3) will allow us to rewrite the capital accumulation equation in terms of capital per worker. But before we proceed, let's first consider the growth rate of the labor force, $\dot{L}/L$. An important assumption that will be maintained throughout most of this book is that the labor force participation rate is constant and that the population growth rate is given by the parameter n.[5] This implies that the labor force growth rate, $\dot{L}/L$, is also given by n. If $n = .01$, then the population and the labor force are growing at one percent per year. This exponential growth can be seen from the relationship

$$L(t) = L_0 e^{nt}.$$

Take logs and differentiate this equation, and what do you get?

[5]Often, it is convenient in describing the model to assume that the labor force participation rate is unity—i.e., every member of the population is also a worker.

Now we are ready to combine Example 1 and equation (2.3):

$$\frac{\dot{k}}{k} = \frac{sY}{K} - n - d$$
$$= \frac{sy}{k} - n - d.$$

This now yields the capital accumulation equation in per worker terms:

$$\dot{k} = sy - (n + d)k.$$

This equation says that the change in capital per worker each period is determined by three terms. Two of the terms are analogous to the original capital accumulation equation. Investment per worker, sy, increases k, while depreciation per worker, dk, reduces k. The term that is new in this equation is a reduction in k because of population growth, the nk term. Each period, there are nL new workers around who were not there during the last period. If there were no new investment and no depreciation, capital *per worker* would decline because of the increase in the labor force. The amount by which it would decline is exactly nk, as can be seen by setting $\dot{K}$ to zero in Example 1.

2.1.1 SOLVING THE BASIC SOLOW MODEL

We have now laid out the basic elements of the Solow model and it is time to begin solving the model. What does it mean to "solve" a model? To answer this question we need to explain exactly what a model is and to define some concepts.

In general, a model consists of several equations that describe the relationships among a collection of *endogenous variables*—that is, among variables whose values are determined within the model itself. For example, equation (2.1) shows how output is produced from capital and labor, and equation (2.3) shows how capital is accumulated over time. Output, Y, and capital, K, are endogenous variables, as are the respective "per worker" versions of these variables, y and k.

Notice that the equations describing the relationships among endogenous variables also involve *parameters* and *exogenous variables*. Parameters are terms such as α, s, k_0, and n that stand in for single numbers. Exogenous variables are terms that may vary over time but whose values are determined outside of the model—i.e., exogenously.

The number of workers in this economy, L, is an example of an exogenous variable.

With these concepts explained, we are ready to tackle the question of what it means to solve a model. Solving a model means obtaining the values of each endogenous variable when given values for the exogenous variables and parameters. Ideally, one would like to be able to express each endogenous variable as a function only of exogenous variables and parameters. Sometimes this is possible; other times a diagram can provide insights into the nature of the solution but a computer is needed for exact values.

For this purpose, it is helpful to think of the economist as a laboratory scientist. The economist sets up a model and has control over the parameters and exogenous variables. The "experiment" is the model itself. Once the model is setup, the economist starts the experiment and watches to see how the endogenous variables evolve over time. The economist is free to vary the parameters and exogenous variables in different experiments to see how this changes the evolution of the endogenous variables.

In the case of the Solow model, our solution will proceed in several steps. We begin with several diagrams that provide insight into the solution. Then, in Section 2.1.4, we provide an analytic solution for the long-run values of the key endogenous variables. A full solution of the model at every point in time is possible analytically, but this derivation is somewhat difficult and is relegated to the appendix of this chapter.

2.1.2 THE SOLOW DIAGRAM

At the beginning of this section we derived the two key equations of the Solow model in terms of output per worker and capital per worker. These equations are

$$y = k^{\alpha} \tag{2.4}$$

and

$$\dot{k} = sy - (n + d)k. \tag{2.5}$$

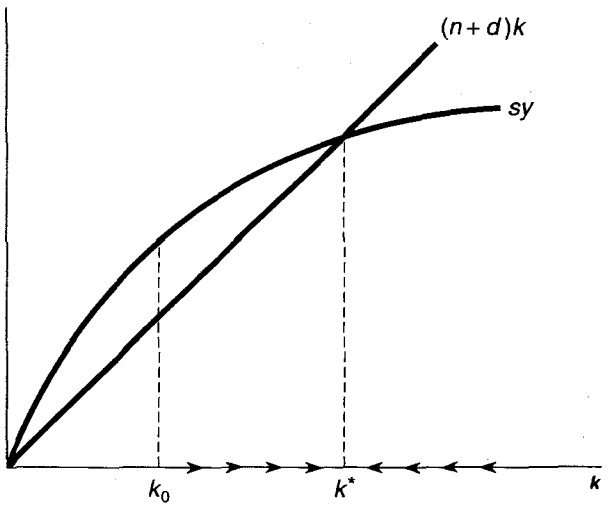
Now we are ready to ask fundamental questions of our model. For example, an economy starts out with a given stock of capital per worker, k_0, and a given population growth rate, depreciation rate, and investment

rate. How does output per worker evolve over time in this economy — i.e., how does the economy grow? How does output per worker compare in the long run between two economies that have different investment rates?

These questions are most easily analyzed in a Solow diagram, as shown in Figure 2.2. The Solow diagram consists of two curves, plotted as functions of the capital-labor ratio, k. The first curve is the amount of investment per person, $sy = sk^{\alpha}$. This curve has the same shape as the production function plotted in Figure 2.1, but it is translated down by the factor s. The second curve is the line $(n + d)k$, which represents the amount of new investment per person required to keep the amount of capital per worker constant — both depreciation and the growing workforce tend to reduce the amount of capital per person in the economy. By no coincidence, the difference between these two curves is the change in the amount of capital per worker. When this change is positive and the economy is increasing its capital per worker, we say that *capital deepening* is occurring. When this per worker change is zero but the actual capital stock K is growing (because of population growth), we say that only *capital widening* is occurring.

To consider a specific example, suppose an economy has capital equal to the amount k_0 today, as drawn in Figure 2.2. What happens over

FIGURE 2.2 **THE BASIC SOLOW DIAGRAM**



time? At k_0, the amount of investment per worker exceeds the amount needed to keep capital per worker constant, so that capital deepening occurs — that is, k increases over time. This capital deepening will continue until $k = k^*$, at which point $sy = (n + d)k$, so that $\dot{k} = 0$. At this point, the amount of capital per worker remains constant, and we call such a point a *steady state*.

What would happen if instead the economy began with a capital stock per worker larger than k^*? At points to the right of k^* in Figure 2.2, the amount of investment per worker provided by the economy is less than the amount needed to keep the capital-labor ratio constant. The term $\dot{k}$ is negative, and therefore the amount of capital per worker begins to decline in this economy. This decline occurs until the amount of capital per worker falls to k^*.

Notice that the Solow diagram determines the steady-state value of capital per worker. The production function of equation (2.4) then determines the steady-state value of output per worker, y^*, as a function of k^*. It is sometimes convenient to include the production function in the Solow diagram itself to make this point clearly. This is done in

FIGURE 2.3 **THE SOLOW DIAGRAM AND THE PRODUCTION FUNCTION**

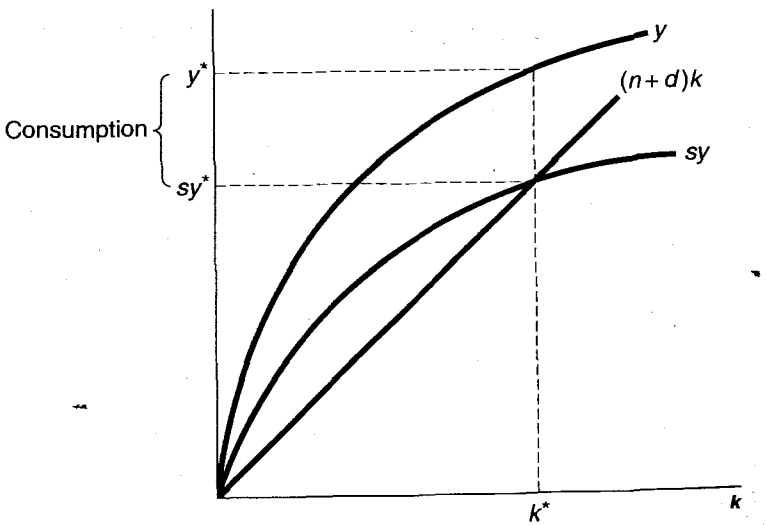


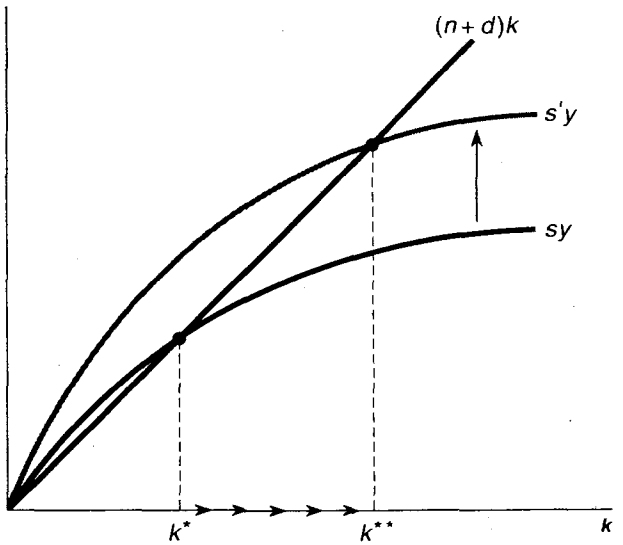
Figure 2.3. Notice that steady-state consumption per worker is then given by the difference between steady-state output per worker, y^*, and steady-state investment per worker, sy^*.

2.1.3 COMPARATIVE STATICS

Comparative statics are used to examine the response of the model to changes in the values of various parameters. In this section, we will consider what happens to per capita income in an economy that begins in steady state but then experiences a "shock." The shocks we will consider are an increase in the investment rate, s, and an increase in the population growth rate, n.

AN INCREASE IN THE INVESTMENT RATE Consider an economy that has arrived at its steady-state value of output per worker. Now suppose that the consumers in that economy decide to increase the investment rate permanently from s to some value s'. What happens to k and y in this economy?

FIGURE 2.4 **AN INCREASE IN THE INVESTMENT RATE**

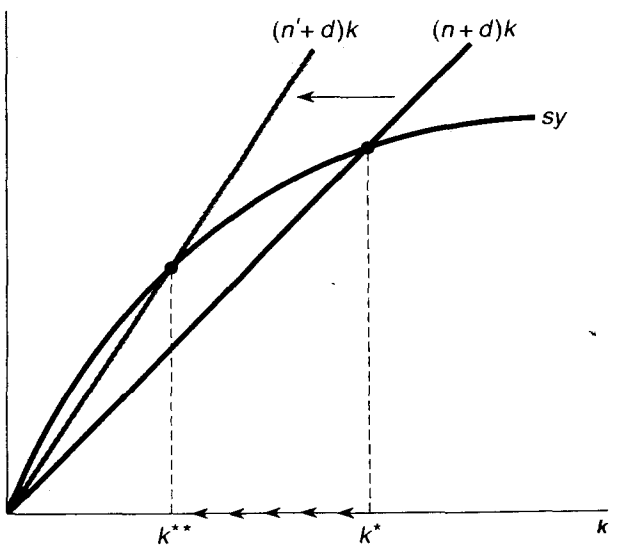


The answer is found in Figure 2.4. The increase in the investment rate shifts the sy curve upward to $s'y$. At the current value of the capital stock, k^*, investment per worker now exceeds the amount required to keep capital per worker constant, and therefore the economy begins capital deepening again. This capital deepening continues until $s'y = (n + d)k$ and the capital stock per worker reaches a higher value, indicated by the point k^{**}. From the production function, we know that this higher level of capital per worker will be associated with higher per capita output; the economy is now richer than it was before.

AN INCREASE IN THE POPULATION GROWTH RATE Now consider an alternative exercise. Suppose an economy has reached its steady state, but then because of immigration, for example, the population growth rate of the economy rises from n to n'. What happens to k and y in this economy?

Figure 2.5 computes the answer graphically. The $(n + d)k$ curve rotates up and to the left to the new curve $(n' + d)k$. At the current value of the capital stock, k^*, investment per worker is now no longer high enough to keep the capital-labor ratio constant in the face of the rising

FIGURE 2.5 **AN INCREASE IN POPULATION GROWTH**



population. Therefore the capital-labor ratio begins to fall. It continues to fall until the point at which $sy = (n' + d)k$, indicated by k^{**} in Figure 2.5. At this point, the economy has less capital per worker than it began with and is therefore poorer: per capita output is ultimately lower after the increase in population growth in this example. Why?

2.1.4 PROPERTIES OF THE STEADY STATE

By definition, the steady-state quantity of capital per worker is determined by the condition that $\dot{k} = 0$. Equations (2.4) and (2.5) allow us to use this condition to solve for the steady-state quantities of capital per worker and output per worker. Substituting from (2.4) into (2.5),

$$\dot{k} = sk^{\alpha} - (n + d)k,$$

and setting this equation equal to zero yields

$$k^* = \left(\frac{s}{n + d}\right)^{1/(1-\alpha)}.$$

Substituting this into the production function reveals the steady-state quantity of output per worker, y^*:

$$y^* = \left(\frac{s}{n + d}\right)^{\alpha/(1-\alpha)}.$$

Notice that the endogenous variable y^* is now written in terms of the parameters of the model. Thus, we have a "solution" for the model, at least in the steady state.

This equation reveals the Solow model's answer to the question "Why are we so rich and they so poor?" Countries that have high savings/investment rates will tend to be richer, *ceteris paribus*.[6] Such countries accumulate more capital per worker, and countries with more capital per worker have more output per worker. Countries that have high population growth rates, in contrast, will tend to be poorer, according to the Solow model. A higher fraction of savings in these economies must go simply to keep the capital-labor ratio constant in the face of a growing population. This capital-widening requirement makes capital

[6] *Ceteris paribus* is Latin for "all other things being equal."

deepening more difficult, and these economies tend to accumulate less capital per worker.

How well do these predictions of the Solow model hold up empirically? Figures 2.6 and 2.7 plot GDP per worker against gross investment as a share of GDP and against population growth rates, respectively. Broadly speaking, the predictions of the Solow model are borne out by the empirical evidence. Countries with high investment rates tend to be richer on average than countries with low investment rates, and countries with high population growth rates tend to be poorer on average. At this level, then, the general predictions of the Solow model seem to be supported by the data.

FIGURE 2.6 GDP PER WORKER VERSUS THE INVESTMENT RATE

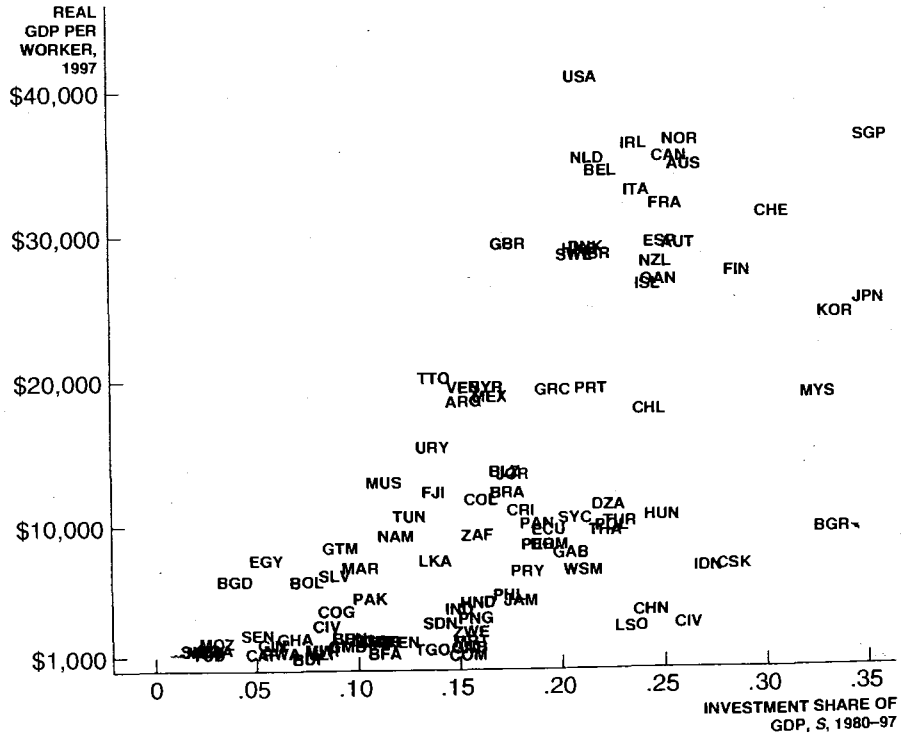
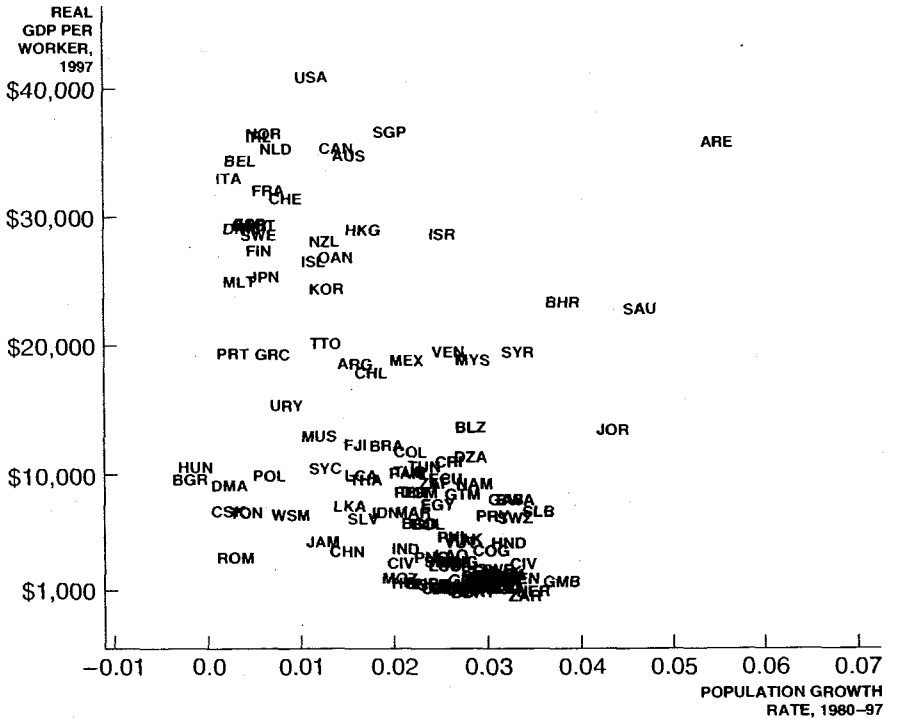


FIGURE 2.7 **GDP PER WORKER VERSUS POPULATION GROWTH RATES**



2.1.5 ECONOMIC GROWTH IN THE SIMPLE MODEL

What does economic growth look like in the steady state of this simple version of the Solow model? The answer is that there is *no* per capita growth in this version of the model! Output per worker (and therefore per person, since we've assumed the labor force participation rate is constant) is constant in the steady state. Output itself, Y, is growing, of course, but only at the rate of population growth.[7]

This version of the model fits several of the stylized facts discussed in Chapter 1. It generates differences in per capita income across countries. It generates a constant capital-output ratio (because both k and y are constant, implying that K/Y is constant). It generates a constant interest rate, the marginal product of capital. However, it fails to predict a very important stylized fact: that economies exhibit sustained per capita income growth. In this model, economies may grow for a while, but not forever. For example, an economy that begins with a stock of capital per worker below its steady-state value will experience growth in k and y along the *transition path* to the steady state. Over time, however, growth slows down as the economy approaches its steady state, and eventually growth stops altogether.

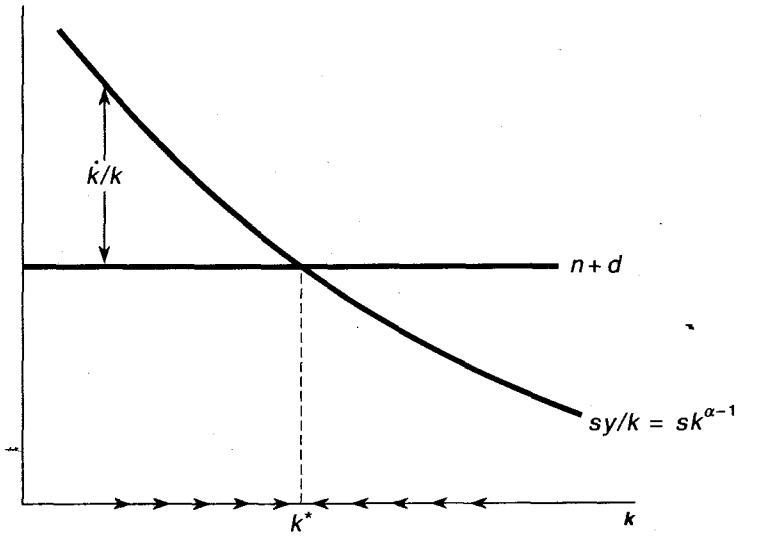
[7]This can be seen easily by applying the "take logs and differentiate" trick to $y \equiv Y/L$.

To see that growth slows down along the transition path, notice two things. First, from the capital accumulation equation (equation (2.5)), one can divide both sides by k to get

$$\frac{\dot{k}}{k} = sk^{\alpha-1} - (n + d). \tag{2.6}$$

Because α is less than one, as k rises, the growth rate of k gradually declines. Second, from Example 2, the growth rate of y is proportional to the growth rate of k, so that the same statement holds true for output per worker.

FIGURE 2.8 **TRANSITION DYNAMICS**



The transition dynamics implied by equation (2.6) are plotted in Figure 2.8.[8] The first term on the right-hand side of the equation is $sk^{\alpha-1}$, which is equal to sy/k. The higher the level of capital per worker, the lower the average product of capital, y/k, because of diminishing returns to capital accumulation (α is less than one). Therefore, this curve slopes downward. The second term on the right-hand side of equation (2.6) is $n + d$, which doesn't depend on k, so it is plotted as a horizontal line. The difference between the two lines in Figure 2.8 is the growth rate of the capital stock, or $\dot{k}/k$. Thus, the figure clearly indicates that the further an economy is below its steady-state value of k, the faster the economy grows. Also, the further an economy is above its steady-state value of k, the faster k declines.

2 TECHNOLOGY AND THE SOLOW MODEL

To generate sustained growth in per capita income in this model, we must follow Solow and introduce technological progress to the model. This is accomplished by adding a technology variable, A, to the production function:

$$Y = F(K, AL) = K^{\alpha}(AL)^{1-\alpha}. \tag{2.7}$$

Entered this way, the technology variable A is said to be "labor-augmenting" or "Harrod-neutral."[9] Technological progress occurs when A increases over time — a unit of labor, for example, is more productive when the level of technology is higher.

An important assumption of the Solow model is that technological progress is *exogenous*: in a common phrase, technology is like "manna from heaven," in that it descends upon the economy automatically and regardless of whatever else is going on in the economy. Instead of modeling carefully where technology comes from, we simply recognize for the moment that there is technological progress and make the assumption that A is growing at a constant rate:

$$\frac{\dot{A}}{A} = g \Longleftrightarrow A = A_0 e^{gt},$$

where g is a parameter representing the growth rate of technology. Of course, this assumption about technology is unrealistic, and explaining how to relax this assumption is one of the major accomplishments of the "new" growth theory that we will explore in later chapters.

The capital accumulation equation in the Solow model with technology is the same as before. Rewriting it slightly, we get

$$\frac{\dot{K}}{K} = s\frac{Y}{K} - d. \tag{2.8}$$

To see the growth implications of the model with technology, first rewrite the production function (2.7) in terms of output per worker:

$$y = k^{\alpha}A^{1-\alpha}.$$

Then take logs and differentiate:

$$\frac{\dot{y}}{y} = \alpha\frac{\dot{k}}{k} + (1-\alpha)\frac{\dot{A}}{A}. \tag{2.9}$$

Finally, notice from the capital accumulation equation (2.8) that the growth rate of K will be constant if and only if Y/K is constant. Furthermore, if Y/K is constant, y/k is also constant, and most important, y and k will be growing at the same rate. A situation in which capital, output, consumption, and population are growing at constant rates is called a *balanced growth path*. Partly because of its empirical appeal, this is a situation that we often wish to analyze in our models. For example, according to Fact 5 in Chapter 1, this situation describes the U.S. economy.

Let's use the notation g_x to denote the growth rate of some variable x along a balanced growth path. Then, along a balanced growth path, $g_y = g_k$ according to the argument above. Substituting this relationship into equation (2.9) and recalling that $\dot{A}/A = g$,

$$g_y = g_k = g. \tag{2.10}$$

That is, along a balanced growth path in the Solow model, output per worker and capital per worker both grow at the rate of exogenous tech-

[8] This alternative version of the Solow diagram makes the growth implications of the Solow model much more transparent. Xavier Sala-i-Martin (1990) emphasizes this point.

[9] The other possibilities are $F(AK, L)$, which is known as "capital-augmenting" or "Solow-neutral" technology, and $AF(K, L)$, which is known as "Hicks-neutral" technology. With the Cobb-Douglas functional form assumed here, this distinction is less important.

nological change, g. Notice that in the model of Section 2.1, there was no technological progress, and therefore there was no long-run growth in output per worker or capital per worker; $g_y = g_k = g = 0$. The model with technology reveals that *technological progress is the source of sustained per capita growth.* In this chapter, this result is little more than an assumption; in later chapters, we will explore the result in much more detail and come to the same conclusion.

2.2.1 THE SOLOW DIAGRAM WITH TECHNOLOGY

The analysis of the Solow model with technological progress proceeds very much like the analysis in Section 2.1: we set up a differential equation and analyze it in a Solow diagram to find the steady state. The only important difference is that the variable k is no longer constant in the long run, so we have to write our differential equation in terms of another variable. The new *state* variable will be $\tilde{k} \equiv K/AL$. Notice that this is equivalent to k/A and is obviously constant along the balanced growth path because $g_k = g_A = g$. The variable $\tilde{k}$ therefore represents the ratio of capital per worker to technology. We will refer to this as the "capital-technology" ratio (keeping in mind that the numerator is capital per worker rather than the total level of capital).

Rewriting the production function in terms of $\tilde{k}$, we get

$$\tilde{y} = \tilde{k}^{\alpha}, \tag{2.11}$$

where $\tilde{y} \equiv Y/AL = y/A$. Following the terminology above, we will refer to $\tilde{y}$ as the "output-technology ratio."[10]

Rewriting the capital accumulation equation in terms of $\tilde{k}$ is accomplished by following exactly the methodology used in Section 2.1. First, note that

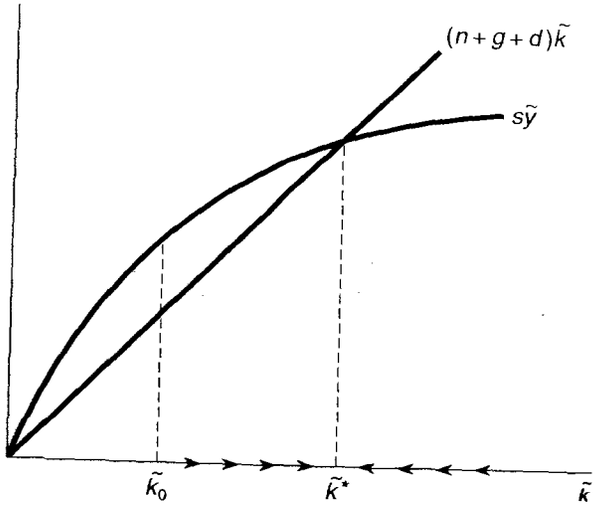
$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L}.$$

Combining this with the capital accumulation equation reveals that

$$\dot{\tilde{k}} = s\tilde{y} - (n + g + d)\tilde{k}. \tag{2.12}$$

[10]The variables $\tilde{y}$ and $\tilde{k}$ are sometimes referred to as "output per effective unit of labor" and "capital per effective unit of labor." This labeling is motivated by the fact that technological progress is labor-augmenting. AL is then the "effective" amount of labor used in production.

FIGURE 2.9 THE SOLOW DIAGRAM WITH TECHNOLOGICAL PROGRESS



The similarity of equations (2.11) and (2.12) to their counterparts in Section 2.1 should be obvious.

The Solow diagram with technological progress is presented in Figure 2.9. The analysis of this diagram is very similar to the analysis when there is no technological progress, but the interpretation is slightly different. If the economy begins with a capital-technology ratio that is below its steady-state level, say at a point such as $\tilde{k}_0$, the capital-technology ratio will rise gradually over time. Why? Because the amount of investment being undertaken exceeds the amount needed to keep the capital-technology ratio constant. This will be true until $s\tilde{y} = (n+g+d)\tilde{k}$ at the point $\tilde{k}^*$, at which point the economy is in steady state and grows along a balanced growth path.

2.2.2 SOLVING FOR THE STEADY STATE

The steady-state output-technology ratio is determined by the production function and the condition that $\dot{\tilde{k}} = 0$. Solving for $\tilde{k}^*$, we find

that

$$\tilde{k}^* = \left(\frac{s}{n+g+d}\right)^{1/(1-\alpha)}.$$

Substituting this into the production function yields

$$\tilde{y}^* = \left(\frac{s}{n+g+d}\right)^{\alpha/(1-\alpha)}.$$

To see what this implies about output per worker, rewrite the equation as

$$y^*(t) = A(t)\left(\frac{s}{n+g+d}\right)^{\alpha/(1-\alpha)}, \qquad (2.13)$$

where we explicitly note the dependence of y and A on time. From equation (2.13), we see that output per worker along the balanced growth path is determined by technology, the investment rate, and the population growth rate. For the special case of $g = 0$ and $A_0 = 1$—i.e., of no technological progress—this result is identical to that derived in Section 2.1.

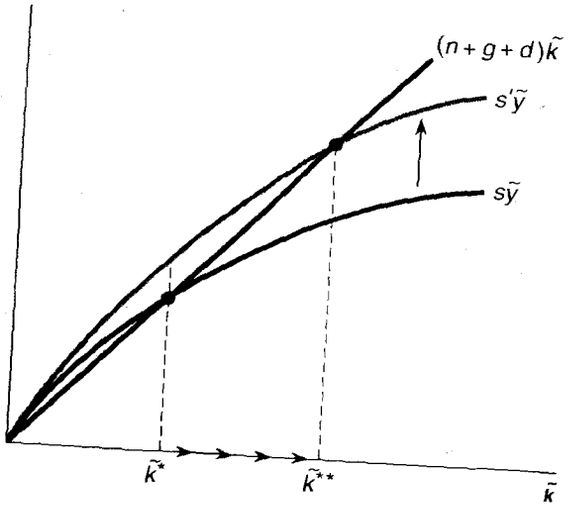
An interesting result is apparent from equation (2.13) and is discussed in more detail in Exercise 1 at the end of this chapter. That is, changes in the investment rate or the population growth rate affect the long-run *level* of output per worker but do not affect the long-run *growth rate* of output per worker. To see this more clearly, let's consider a simple example.

Suppose an economy begins in steady state with investment rate s and then permanently increases its investment rate to s' (for example, because of a permanent subsidy to investment). The Solow diagram for this policy change is drawn in Figure 2.10, and the results are broadly similar to the case with no technological progress. At the initial capital-technology ratio $\tilde{k}^*$, investment exceeds the amount needed to keep the capital-technology ratio constant, so $\tilde{k}$ begins to rise.

To see the effects on growth, rewrite equation (2.12) as

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = s\frac{\tilde{y}}{\tilde{k}} - (n+g+d),$$

FIGURE 2.10 **AN INCREASE IN THE INVESTMENT RATE**



and note that $\tilde{y}/\tilde{k}$ is equal to $\tilde{k}^{\alpha-1}$. Figure 2.11 illustrates the transition dynamics implied by this equation. As the diagram shows, the increase in the investment rate to s' raises the growth rate temporarily as the economy transits to the new steady state, $\tilde{k}^{**}$. Since g is constant, faster growth in $\tilde{k}$ along the transition path implies that output per worker increases more rapidly than technology: $\dot{y}/y > g$. The behavior of the growth rate of output per worker over time is displayed in Figure 2.12.

Figure 2.13 cumulates the effects on growth to show what happens to the (log) level of output per worker over time. Prior to the policy change, output per worker is growing at the constant rate g, so that the log of output per worker rises linearly. At the time of the policy change, t^*, output per worker begins to grow more rapidly. This more rapid growth continues temporarily until the output-technology ratio reaches its new steady state. At this point, growth has returned to its long-run level of g.

This exercise illustrates two important points. First, policy changes in the Solow model increase growth rates, but only temporarily along the transition to the new steady state. That is, policy changes have no long-run *growth effect*. Second, policy changes can have *level effects*.

FIGURE 2.11 AN INCREASE IN THE INVESTMENT RATE: TRANSITION DYNAMICS

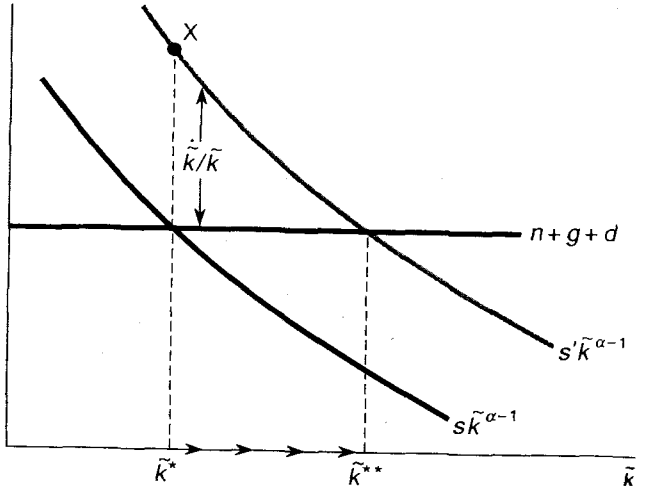


FIGURE 2.12 THE EFFECT OF AN INCREASE IN INVESTMENT ON GROWTH

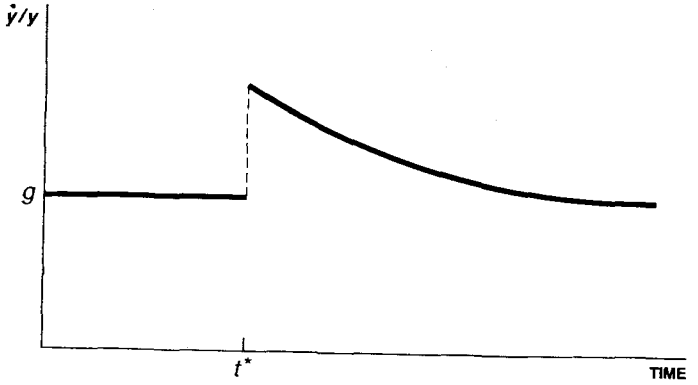
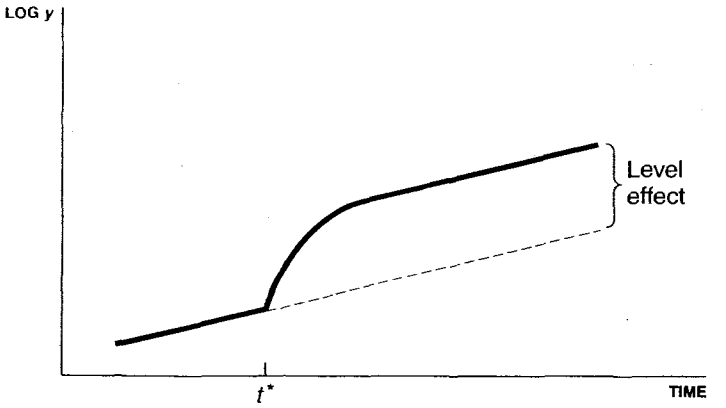


FIGURE 2.13 THE EFFECT OF AN INCREASE IN INVESTMENT ON y



That is, a permanent policy change can permanently raise (or lower) the level of per capita output.

2.3 EVALUATING THE SOLOW MODEL

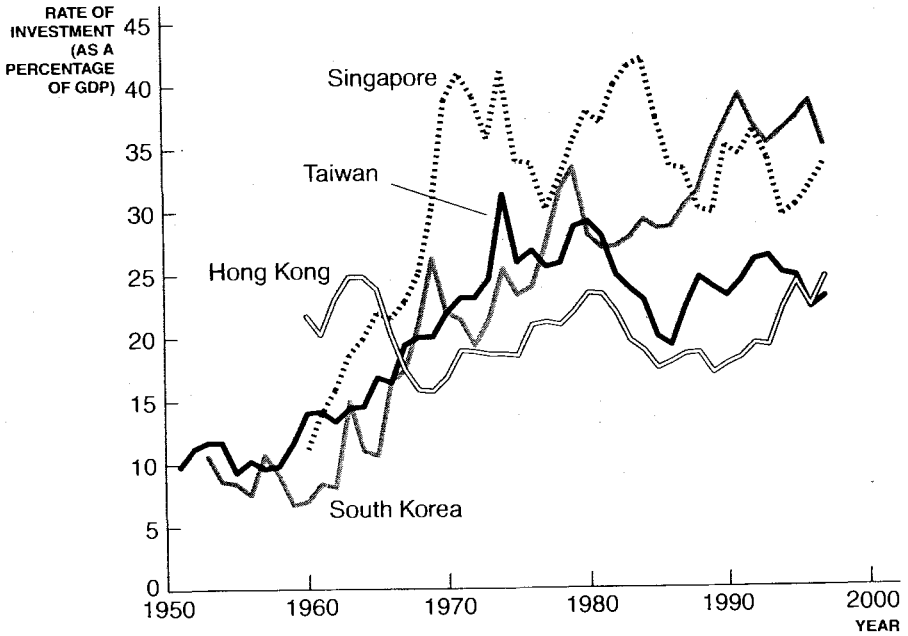
How does the Solow model answer the key questions of growth and development? First, the Solow model appeals to differences in investment rates and population growth rates and (perhaps) to exogenous differences in technology to explain differences in per capita incomes. Why are we so rich and they so poor? According to the Solow model, it is because we invest more and have lower population growth rates, both of which allow us to accumulate more capital per worker and thus increase labor productivity. In the next chapter, we will explore this hypothesis more carefully and see that it is firmly supported by data across the countries of the world.

Second, why do economies exhibit sustained growth in the Solow model? The answer is technological progress. As we saw earlier, without technological progress, per capita growth will eventually cease as diminishing returns to capital set in. Technological progress, however, can offset the tendency for the marginal product of capital to fall, and

in the long run, countries exhibit per capita growth at the rate of technological progress.

How, then, does the Solow model account for differences in growth rates across countries? At first glance, it may seem that the Solow model cannot do so, except by appealing to differences in (unmodeled) technological progress. A more subtle explanation, however, can be found by appealing to transition dynamics. We have seen several examples of how transition dynamics can allow countries to grow at rates different from their long-run growth rates. For example, an economy with a capital-technology ratio below its long-run level will grow rapidly until the capital-technology ratio reaches its steady-state level. This reasoning may help explain why countries such as Japan and Germany, which had their capital stocks wiped out by World War II, have grown more rapidly than the United States over the last fifty years. Or it may explain why an economy that increases its investment rate will grow rapidly as

FIGURE 2.14 **INVESTMENT RATES IN SOME NEWLY INDUSTRIALIZING ECONOMIES**



it makes the transition to a higher output-technology ratio. This explanation may work well for countries such as South Korea, Singapore, and Taiwan. Their investment rates have increased dramatically since 1950, as shown in Figure 2.14. The explanation may work less well, however, for an economy such as Hong Kong's. This kind of reasoning raises an interesting question: can countries permanently grow at different rates? This question will be discussed in more detail in later chapters.

GROWTH ACCOUNTING, THE PRODUCTIVITY SLOWDOWN, AND THE NEW ECONOMY

We have seen in the Solow model that sustained growth occurs only in the presence of technological progress. Without technological progress, capital accumulation runs into diminishing returns. With technological progress, however, improvements in technology continually offset the diminishing returns to capital accumulation. Labor productivity grows as a result, both directly because of the improvements in technology and indirectly because of the additional capital accumulation these improvements make possible.

In 1957, Solow published a second article, "Technical Change and the Aggregate Production Function," in which he performed a simple accounting exercise to break down growth in output into growth in capital, growth in labor, and growth in technological change. This "growth-accounting" exercise begins by postulating a production function such as

$$Y = BK^{\alpha}L^{1-\alpha},$$

where B is a Hicks-neutral productivity term.[11] Taking logs and differentiating this production function, one derives the key formula of growth accounting:

$$\frac{\dot{Y}}{Y} = \alpha\frac{\dot{K}}{K} + (1-\alpha)\frac{\dot{L}}{L} + \frac{\dot{B}}{B}. \qquad (2.14)$$

[11] In fact, this growth accounting can be done with a much more general production function such as $B(t)F(K, L)$, and the results are very similar.

This equation says that output growth is equal to a weighted average of capital and labor growth plus the growth rate of B. This last term, $\dot{B}/B$, is commonly referred to as *total factor productivity growth* or *multifactor productivity growth*. Solow, as well as economists such as Edward Denison and Dale Jorgenson who followed Solow's approach, have used this equation to understand the sources of growth in output.

Since we are primarily interested here in the growth rate of output per worker instead of total output, it is helpful to rewrite equation (2.14) by subtracting $\dot{L}/L$ from both sides:

$$\frac{\dot{y}}{y} = \alpha\frac{\dot{k}}{k} + \frac{\dot{B}}{B}. \tag{2.15}$$

That is, the growth rate of output per worker is decomposed into the contribution of physical capital per worker and the contribution from multifactor productivity growth.

The U.S. Bureau of Labor Statistics (BLS) provides a detailed accounting of U.S. growth using a generalization of equation (2.15). Its most recent numbers are reported in Table 2.1. They generalize this equation in a couple of ways. First, the BLS measures labor by calculating total hours worked rather than just the number of workers. Second, the BLS includes an additional term in equation (2.15) to adjust for the changing composition of the labor force—to recognize, for example, that the labor force is more educated today than it was forty years ago.

TABLE 2.1 GROWTH ACCOUNTING FOR THE UNITED STATES

	1948–98	48–73	73–79	79–90	90–95	95–98
Output per hour	2.5	3.3	1.3	1.6	1.5	2.5
Contributions from:						
Capital per hour worked	0.8	1.0	0.7	0.7	0.5	0.8
Information technology	0.3	0.1	0.3	0.5	0.4	0.8
Other capital services	0.6	0.9	0.5	0.3	0.1	0.0
Labor composition	0.2	0.2	0.0	0.3	0.4	0.3
Multifactor productivity	1.4	2.1	0.6	0.5	0.6	1.4

SOURCE: Bureau of Labor Statistics (2000).
Note: The table reports average annual growth rates for the private business sector. "Information technology" refers to information processing equipment and software.

As can be seen from the table, output per hour in the private business sector for the United States grew at an average annual rate of 2.5 percent between 1948 and 1998. The contribution from capital per hour worked was 0.8 percentage points, and the changing composition of the labor force contributed another 0.2 percentage points. Multifactor productivity growth accounts for the remaining 1.4 percentage points, by definition. The implication is that about one-half of U.S. growth was due to factor accumulation and one-half was due to the improvement in the productivity of these factors over this period. Because of the way in which it is calculated, economists have referred to this 1.4 percent as the "residual" or even as a "measure of our ignorance." One interpretation of the multifactor productivity growth term is that it is due to technological change; notice that in terms of the production function in equation (2.7), $B = A^{1-\alpha}$. This interpretation will be explored in later chapters.

Table 2.1 also reveals how GDP growth and its sources have changed over time in the United States. One of the important stylized facts revealed in the table is the productivity growth slowdown that occurred in the 1970s. The top row shows that growth in output per hour (also known as labor productivity) slowed dramatically after 1973; growth between 1973 and 1995 was nearly 2 percentage points slower than growth between 1948 and 1973. What was the source of this slowdown? The next few rows show that the changes in the contributions from capital per worker and labor composition are relatively minor. The primary culprit of the productivity slowdown is a substantial decline in the growth rate of multifactor productivity. For some reason, the "residual" was much lower after 1973 than before: the bulk of the productivity slowdown is accounted for by the "measure of our ignorance." A similar productivity slowdown occurred throughout the advanced countries of the world.

Various explanations for the productivity slowdown have been advanced. For example, perhaps the sharp rise in energy prices in 1973 and 1979 contributed to the slowdown. One problem with this explanation is that in real terms energy prices were lower in the late 1980s

than they were before the oil shocks. Another explanation may involve the changing composition of the labor force or the sectoral shift in the economy away from manufacturing (which tends to have high labor productivity) toward services (many of which have low labor productivity). This explanation receives some support from recent evidence that productivity growth recovered substantially in the 1980s in manufacturing. It is possible that a slowdown in resources spent on research and development (R&D) in the late 1960s contributed to the slowdown as well. Or, perhaps it is not the 1970s and 1980s that need to be explained but rather the 1950s and 1960s: growth may simply have been artificially and temporarily high in the years following World War II because of the application to the private sector of new technologies created for the war. Nevertheless, careful work on the productivity slowdown has failed to provide a complete explanation.[12]

The flip side of the productivity slowdown after 1973 is the rise in productivity growth in the 1995–98 period, sometimes labeled the "New Economy." Growth in output per hour and in multifactor productivity rose substantially in this period, returning about 50 percent of the way back to the growth rates exhibited before 1973. As shown in Table 2.1, the increase in growth rates is partially associated with an increase in the use of information technology. Before 1973, this component of capital accumulation contributed only 0.1 percentage points of growth, but by the late 1990s, this contribution had risen to 0.8 percentage points. In addition, evidence suggests that as much as half of the rise in multifactor productivity growth in recent years is due to increases in efficiency of the production of information technology.

Recently, a number of economists have suggested that the information-technology revolution associated with the widespread adoption of computers might explain both the productivity slowdown after 1973 as well as the recent rise in productivity growth. According to this hypothesis, growth slowed temporarily while the economy adapted its factories to the new production techniques associated with information technology and as workers learned to take advantage of the new technology. The recent upsurge in productivity growth, then, reflects the

[12]The fall 1988 issue of the *Journal of Economic Perspectives* contains several papers discussing potential explanations of the productivity slowdown.

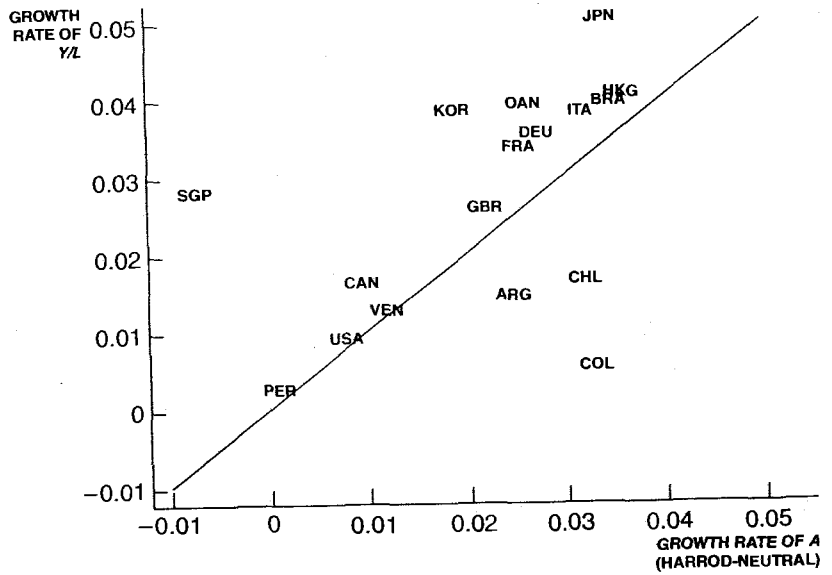
successful widespread adoption of this new technology. The recent upsurge in productivity growth, then, reflects the successful widespread adoption of this new technology.[13] Whether or not this view is correct remains to be seen.

Growth accounting has also been used to analyze economic growth in countries other than the United States. One of the more interesting applications is to the NICs of South Korea, Hong Kong, Singapore, and Taiwan. Recall from Chapter 1 that average annual growth rates have exceeded 5 percent in these economies since 1960. Alwyn Young (1995) shows that a large part of this growth is the result of factor accumulation: increases in investment in physical capital and education, increases in labor force participation, and a shift from agriculture into manufacturing. Support for Young's result is provided in Figure 2.15. The vertical axis measures growth in output per worker, while the horizontal axis measures growth in Harrod-neutral (i.e., labor-augmenting) total factor productivity. That is, instead of focusing on growth in B, where $B = A^{1-\alpha}$, we focus on the growth of A. (Notice that with $\alpha = \frac{1}{3}$, the growth rate of A is simply 1.5 times the growth rate of B.) This change of variables is often convenient because along a steady-state balanced growth path, $g_y = g_A$. Countries growing along a balanced growth path, then, should lie on the 45-degree line in the figure.

Two features of Figure 2.15 stand out. First, while the growth rates of output per worker in the East Asian countries are clearly remarkable, their rates of growth in total factor productivity (TFP) are less so. A number of other countries such as Italy, Brazil, and Chile have also experienced rapid TFP growth. Total factor productivity growth, while typically higher than in the United States, was not exceptional in the East Asian economies. Second, the East Asian countries are far above the 45-degree line. This shift means that growth in output per worker is much higher than TFP growth would suggest. Singapore is an extreme example, with slightly *negative* TFP growth. Its rapid growth of output per worker is entirely attributable to growth in capital and education. More generally, a key source of the rapid growth performance

[13]See Paul David (1990) and Jeremy Greenwood and Mehmet Yorukoglu (1997). More generally, a nice collection of papers on the "New Economy" can be found in the Fall 2000 issue of the *Journal of Economic Perspectives*.

FIGURE 2.15 **GROWTH ACCOUNTING**



SOURCE: Author's calculations using the data collection reported in Table 10.8 of Barro and Sala-i-Martin (1998).

Note: The years over which growth rates are calculated vary across countries: 1960–90 for OECD members, 1940–80 for Latin America, and 1966–90 for East Asia.

of these countries is factor accumulation. Therefore, Young concludes, the framework of the Solow model (and the extension of the model in Chapter 3) can explain a substantial amount of the rapid growth of the East Asian economies.

APPENDIX: CLOSED-FORM SOLUTION OF THE SOLOW MODEL

It is possible to solve analytically for output per worker $y(t)$ at each point in time in the Solow model. The derivation of this solution is beyond the scope of this book. One derivation can be found in the appendix to Chapter 1 of Barro and Sala-i-Martin (1998). Another can be found in "A Note on the Closed-Form Solution of the Solow Model," which can be downloaded from my Web page at http://emlab.berkeley.edu/users/chad/papers.html#closed form. The key insight is to recognize that the differential equation for the capital-output ratio in the Solow model is linear and can be solved using standard techniques.

Although the method of solution is beyond the scope of this book, the exact solution is still of interest. It illustrates nicely what it means to "solve" a model:

$$y(t) = \left(\frac{s}{n+g+d}(1 - e^{-\lambda t}) + \left(\frac{y_0}{A_0} \right)^{\frac{1-\alpha}{\alpha}} e^{-\lambda t} \right)^{\frac{\alpha}{1-\alpha}} A(t).$$

In this expression, we have defined a new parameter: $\lambda \equiv (1 - \alpha)(n + g + d)$. Notice that output per worker at any time t is written as a function of the parameters of the model as well as of the exogenous variable $A(t)$.

To interpret this expression, notice that at $t = 0$, output per worker is simply equal to y_0, which in turn is given by the parameters of the model; recall that $y_0 = k_0^{\alpha} A_0^{1-\alpha}$. That's a good thing: our solution says that output per worker starts at the level given by the production function! At the other extreme, consider what happens as t gets very large, in the limit going off to infinity. In this case, $e^{-\lambda t}$ goes to zero, so we are left with an expression that is exactly that given by equation (2.13): output per worker reaches its steady-state value.

In between $t = 0$ and $t = \infty$, output per worker is some kind of weighted average of its initial value and its steady-state value. As time goes on, all that changes are the weights.

The interested reader will find it very useful to go back and reinterpret the Solow diagram and the various comparative static exercises with this solution in mind.

EXERCISES

1. *A decrease in the investment rate.* Suppose the U.S. Congress enacts legislation that discourages saving and investment, such as the elimination of the investment tax credit that occurred in 1990. As a result, suppose the investment rate falls permanently from s' to s''.

Examine this policy change in the Solow model with technological progress, assuming that the economy begins in steady state. Sketch a graph of how (the natural log of) output per worker evolves over time with and without the policy change. Make a similar graph for the growth rate of output per worker. Does the policy change permanently reduce the *level* or the *growth rate* of output per worker?

2. *An increase in the labor force.* Shocks to an economy, such as wars, famines, or the unification of two economies, often generate large one-time flows of workers across borders. What are the short-run and long-run effects on an economy of a one-time permanent increase in the stock of labor? Examine this question in the context of the Solow model with $g = 0$ and $n > 0$.

3. *An income tax.* Suppose the U.S. Congress decides to levy an income tax on both wage income and capital income. Instead of receiving $wL + rK = Y$, consumers receive $(1 - \tau)wL + (1 - \tau)rK = (1 - \tau)Y$. Trace the consequences of this tax for output per worker in the short and long runs, starting from steady state.

4. *Manna falls faster.* Suppose that there is a permanent increase in the rate of technological progress, so that g rises to g'. Sketch a graph of the growth rate of output per worker over time. Be sure to pay close attention to the transition dynamics.

5. *Can we save too much?* Consumption is equal to output minus investment: $c = (1 - s)y$. In the context of the Solow model with no technological progress, what is the savings rate that maximizes steady-state consumption per worker? What is the marginal product of capital in this steady state? Show this point in a Solow diagram. Be sure to draw the production function on the diagram, and show consumption and saving and a line indicating the marginal product of capital. Can we save too much?

6. *Solow (1956) versus Solow (1957).* In the Solow model with technological progress, consider an economy that begins in steady state with a rate of technological progress, g, of 2 percent. Suppose g rises permanently to 3 percent. Assume $\alpha = 1/3$.

   (a) What is the growth rate of output per worker before the change, and what happens to this growth rate in the long run?

   (b) Using equation (2.15), perform the growth accounting exercise for this economy, both before the change and after the economy has reached its new balanced growth path. (Hint: recall that $B \equiv A^{1-\alpha}$.) How much of the increase in the growth rate of output per worker is due to a change in the growth rate of capital per worker, and how much is due to a change in multifactor productivity growth?

   (c) In what sense does the growth accounting result in part (b) produce a misleading picture of this experiment?