To maximize $N(t)$, the value of t must be chosen such that $N'(t) = 0$. This first derivative is

$$N'(t) = V'(t)e^{-rt} - r\left[V(t) + \frac{s}{r}\right]e^{-rt} \qquad \text{[product rule]}$$
$$= [V'(t) - rV(t) - s]e^{-rt}$$

and it will be zero if and only if

$$V'(t) = rV(t) + s$$

Thus, this last equation may be taken as the necessary optimization condition for the choice of the time of sale $\bar{t}$.

The economic interpretation of this condition appeals easily to intuitive reasoning: $V'(t)$ represents the rate of change of the sale value, or the increment in V, if sale is postponed for a year, while the two terms on the right indicate, respectively, the increments in the interest cost and the storage cost entailed by such a postponement of sale (revenue and cost are both reckoned at time $\bar{t}$). So, the idea of the equating of the two sides is to us just some "old wine in a new bottle," for it is nothing but the same MC = MR condition in a different guise!

Present Value of a Perpetual Flow

If a cash flow were to persist forever—a situation exemplified by the interest from a perpetual bond or the revenue from an indestructible capital asset such as land—the present value of the flow would be

$$\Pi = \int_0^{\infty} R(t)e^{-rt}\,dt$$

which is an improper integral.

Example 8 Find the present value of a perpetual income stream flowing at the uniform rate of D dollars per year, if the continuous rate of discount is r. Since, in evaluating an improper integral, we simply take the limit of a proper integral, the result in (13.12) can still be of help. Specifically, we can write

$$\Pi = \int_0^{\infty} De^{-rt}\,dt = \lim_{y\to\infty}\int_0^{y} De^{-rt}\,dt = \lim_{y\to\infty}\frac{D}{r}(1 - e^{-ry}) = \frac{D}{r}$$

Note that the y parameter (number of years) has disappeared from the final answer. This is as it should be, for here we are dealing with a *perpetual* flow. You may also observe that our result (present value = rate of revenue ÷ rate of discount) corresponds precisely to the familiar formula for the so-called "capitalization" of an asset with a perpetual yield.

EXERCISE 13.5

1 Given the following marginal-revenue functions:
(*a*) $R'(Q) = 28Q - e^{0.3Q}$ (*b*) $R'(Q) = 10(1 + Q)^{-2}$
find in each case the total-revenue function $R(Q)$. What initial condition can you introduce to definitize the constant of integration?

2 (*a*) Given the marginal propensity to import $M'(Y) = 0.1$ and the i[nformation that] $M = 20$ when $Y = 0$, find the import function $M(Y)$.
(*b*) Given the marginal propensity to consume $C'(Y) = 0.8 + 0.1Y^{-1/2}$ and the information that $C = Y$ when $Y = 100$, find the consumption function $C(Y)$.

3 Assume that the rate of investment is described by the function $I(t) = 12t^{1/3}$ and that $K(0) = 25$:
(*a*) Find the time path of capital stock K.
(*b*) Find the amount of capital accumulation during the time intervals [0, 1] and [1, 3], respectively.

4 Given a continuous income stream at the constant rate of \$1000 per year:
(*a*) What will be the present value Π if the income stream lasts for 2 years and the continuous discount rate is 0.05 per year?
(*b*) What will be the present value Π if the income stream terminates after exactly 3 years and the discount rate is 0.04?

5 What is the present value of a perpetual cash flow of:
(*a*) \$1450 per year, discounted at $r = 5\%$?
(*b*) \$2460 per year, discounted at $r = 8\%$?

13.6 DOMAR GROWTH MODEL

In the population-growth problem of (13.1) and (13.2) and the capital-formation problem of (13.10), the common objective is to delineate a time path on the basis of some given pattern of change of a variable. In the classic growth model of Professor Domar,* on the other hand, the idea is to stipulate the type of time path required to prevail if a certain equilibrium condition of the economy is to be satisfied.

The Framework

The basic premises of the Domar model are as follows:

1. Any change in the rate of investment flow per year $I(t)$ will produce a dual effect: it will affect the aggregate demand as well as the productive capacity of the economy.
2. The demand effect of a change in $I(t)$ operates through the multiplier process, assumed to work instantaneously. Thus an increase in $I(t)$ will raise the rate of income flow per year $Y(t)$ by a multiple of the increment in $I(t)$. The multiplier is $k = 1/s$, where s stands for the given (constant) marginal propensity to save. On the assumption that $I(t)$ is the only (parametric) expenditure flow that influences the rate of income flow, we can then state

* Evsey D. Domar, "Capital Expansion, Rate of Growth, and Employment," *Econometrica*, April, 1946, pp. 137–147; reprinted in Domar, *Essays in the Theory of Economic Growth*, Oxford University Press, Fair Lawn, N.J., 1957, pp. 70–82.

that

$$\frac{dY}{dt} = \frac{dI}{dt}\frac{1}{s} \tag{13.13}$$

3. The capacity effect of investment is to be measured by the change in the rate of *potential* output the economy is capable of producing. Assuming a constant capacity-capital ratio, we can write

$$\frac{\kappa}{K} \equiv \rho \qquad (= \text{a constant})$$

where κ (the Greek letter kappa) stands for capacity or potential output flow per year, and ρ (the Greek letter rho) denotes the given capacity-capital ratio. This implies, of course, that with a capital stock $K(t)$ the economy is potentially capable of producing an annual product, or income, amounting to $\kappa \equiv \rho K$ dollars. Note that, from $\kappa \equiv \rho K$ (the production function), it follows that $d\kappa = \rho\, dK$, and

$$\frac{d\kappa}{dt} = \rho\frac{dK}{dt} = \rho I \tag{13.14}$$

In Domar's model, equilibrium is defined to be a situation in which productive capacity is fully utilized. To have equilibrium is, therefore, to require the aggregate demand to be exactly equal to the potential output producible in a year; that is, $Y = \kappa$. If we start initially from an equilibrium situation, however, the requirement will reduce to the balancing of the respective *changes* in capacity and in aggregate demand; that is,

$$\frac{dY}{dt} = \frac{d\kappa}{dt} \tag{13.15}$$

What kind of time path of investment $I(t)$ can satisfy this equilibrium condition at all times?

Finding the Solution

To answer this question, we first substitute (13.13) and (13.14) into the equilibrium condition (13.15). The result is the following differential equation:

$$\frac{dI}{dt}\frac{1}{s} = \rho I \qquad \text{or} \qquad \frac{1}{I}\frac{dI}{dt} = \rho s \tag{13.16}$$

Since (13.16) specifies a definite pattern of change for I, we should be able to find the equilibrium (or required) investment path from it.

In this simple case, the solution is obtainable by directly integrating both sides of the second equation in (13.16) with respect to t. The fact that the two sides are identical in equilibrium assures the equality of their integrals. Thus,

$$\int \frac{1}{I}\frac{dI}{dt}\,dt = \int \rho s\, dt$$

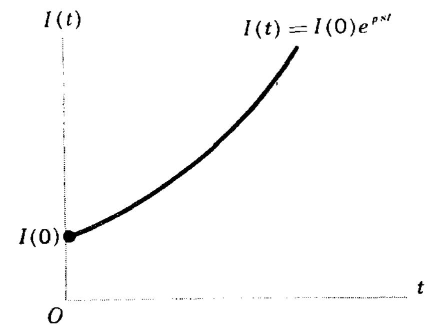

Figure 13.6

By the substitution rule and the log rule, the left side gives us

$$\int \frac{dI}{I} = \ln|I| + c_1 \qquad (I \neq 0)$$

whereas the right side yields (ρs being a constant)

$$\int \rho s\, dt = \rho s t + c_2$$

Equating the two results and combining the two constants, we have

$$\ln|I| = \rho s t + c \tag{13.17}$$

To obtain $|I|$ from $\ln|I|$, we perform an operation known as "taking the antilog of $\ln|I|$," which utilizes the fact that $e^{\ln x} = x$. Thus, letting each side of (13.17) become the exponent of the constant e, we obtain

$$e^{\ln|I|} = e^{(\rho s t + c)}$$

or $\quad |I| = e^{\rho s t}e^{c} = Ae^{\rho s t} \qquad$ where $A \equiv e^{c}$

If we take investment to be positive, then $|I| = I$, so that the above result becomes $I(t) = Ae^{\rho s t}$, where A is arbitrary. To get rid of this arbitrary constant, we set $t = 0$ in the equation $I(t) = Ae^{\rho s t}$, to get $I(0) = Ae^0 = A$. This definitizes the constant A, and enables us to express the solution—the required investment path—as

$$I(t) = I(0)e^{\rho s t} \tag{13.18}$$

where $I(0)$ denotes the initial rate of investment.*

This result has a disquieting economic meaning. In order to maintain the balance between capacity and demand over time, the rate of investment flow must grow precisely at the exponential rate of ρs, along a path such as illustrated in Fig. 13.6. Obviously, the larger will be the required rate of growth of investment,

* The solution (13.18) will remain valid even if we let investment be negative in the result $|I| = Ae^{\rho s t}$. See Exercise 13.6-3.

the larger the capacity-capital ratio and the marginal propensity to save happen to be. But at any rate, once the values of ρ and s are known, the required growth path of investment becomes very rigidly set.

The Razor's Edge

It now becomes relevant to ask what will happen if the *actual* rate of growth of investment—call that rate r—differs from the *required* rate ρs.

Domar's approach is to define a *coefficient of utilization*

$$u = \lim_{t \to \infty} \frac{Y(t)}{\kappa(t)} \qquad [u = 1 \text{ means full utilization of capacity}]$$

and show that $u = r/\rho s$, so that $u \gtreqless 1$ as $r \gtreqless \rho s$. In other words, if there is a discrepancy between the actual and required rates ($r \neq \rho s$), we will find in the end (as $t \to \infty$) either a shortage of capacity ($u > 1$) or a surplus of capacity ($u < 1$), depending on whether r is greater or less than ρs.

We can show, however, that the conclusion about capacity shortage and surplus really applies at any time t, not only as $t \to \infty$. For a growth rate of r implies that

$$I(t) = I(0)e^{rt} \qquad \text{and} \qquad \frac{dI}{dt} = rI(0)e^{rt}$$

Therefore, by (13.13) and (13.14), we have

$$\frac{dY}{dt} = \frac{1}{s}\frac{dI}{dt} = \frac{r}{s}I(0)e^{rt}$$

$$\frac{d\kappa}{dt} = \rho I(t) = \rho I(0)e^{rt}$$

The ratio between these two derivatives,

$$\frac{dY/dt}{d\kappa/dt} = \frac{r}{\rho s}$$

should tell us the relative magnitudes of the demand-creating effect and the capacity-generating effect of investment at any time t, under the actual growth rate of r. If r (the actual rate) exceeds ρs (the required rate), then $dY/dt > d\kappa/dt$, and the demand effect will outstrip the capacity effect, causing a shortage of capacity. Conversely, if $r < \rho s$, there will be a deficiency in aggregate demand and, hence, a surplus of capacity.

The curious thing about this conclusion is that if investment actually grows at a *faster* rate than required ($r > \rho s$), the end result will be a *shortage* rather than a surplus of capacity. It is equally curious that if the actual growth of investment lags behind the required rate ($r < \rho s$), we will encounter a capacity *surplus* rather than a shortage. Indeed, because of such paradoxical results, if we now allow the entrepreneurs to adjust the actual growth rate r (hitherto taken to be a constant) according to the prevailing capacity situation, they will most certainly make the

"wrong" kind of adjustment. In the case of $r > \rho s$, for instance, the emergent capacity shortage will motivate an even faster rate of investment. But this would mean an increase in r, instead of the reduction called for under the circumstances. Consequently, the discrepancy between the two rates of growth would be intensified rather than reduced.

The upshot is that, given the parametric constants ρ and s, the only way to avoid both shortage and surplus of productive capacity is to guide the investment flow ever so carefully along the equilibrium path with a growth rate $\bar{r} = \rho s$. And, as we have shown, any deviation from such a "razor's edge" time path will bring about a persistent failure to satisfy the norm of full utilization which Domar envisaged in this model. This is perhaps not too cheerful a prospect to contemplate. Fortunately, more flexible results become possible when certain assumptions of the Domar model are modified, as we shall see from the growth model of Professor Solow, to be discussed in the next chapter.

EXERCISE 13.6

1 How many factors of production are explicitly considered in the Domar model? What does this fact imply with regard to the capital-labor ratio in production?

2 We learned in Sec. 10.2 that the constant r in the exponential function Ae^{rt} represents the rate of growth of the function. Apply this to (13.16), and deduce (13.18) without going through integration.

3 Show that even if we let investment be negative in the equation $|I| = Ae^{\rho st}$, upon definitizing the arbitrary constant A we will still end up with the solution (13.18).

4 Show that the result in (13.18) can be obtained alternatively by finding—and equating—the *definite* integrals of both sides of (13.16),

$$\frac{1}{I}\frac{dI}{dt} = \rho s$$

with respect to the variable t, with limits of integration $t = 0$ and $t = t$. Remember that when we change the variable of integration from t to I, the limits of integration will change from $t = 0$ and $t = t$, respectively, to $I = I(0)$ and $I = I(t)$.